

Functional Itô calculus in Hilbert spaces and application to path-dependent Kolmogorov equations

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Abstract

Recently, in [4, 5, 13], functional Itô calculus has been introduced and developed in finite dimension for functionals of continuous semimartingales. With different techniques, we develop a functional Itô calculus for functionals of Hilbert space-valued diffusions. In this context, we first prove a path-dependent Itô's formula, then we show applications to classical solutions of path-dependent Kolmogorov equations in Hilbert spaces and derive a Clark-Ocone type formula. Finally, we explicitly verify that all the theory developed can be applied to a class of diffusions driven by SDEs with a path-dependent drift (suitably regular) and constant diffusion coefficient.

Key words: functional Itô calculus, Itô's formula, path-dependent Kolmogorov equation, path-dependent stochastic differential equations, Clark-Ocone formula.

AMS 2010 subject classification: 60H30, 34K50, 35K10, 35R10, 35R15.

1 Introduction

The present paper extends to infinite dimensional spaces the so called functional Itô calculus, so far developed in finite-dimensional spaces, and some of its applications.

In [13] the first ideas for a functional Itô calculus were presented for one-dimensional continuous semimartingales, by introducing suitable notions of time/space derivatives which reveal to be adequate for dealing with non-anticipative functionals. In that paper, a functional Itô's formula is provided and then employed to represent solutions of a backward Kolmogorov equation with path-dependent terminal value. This allows to obtain an explicit representation of the stochastic integrand in the martingale representation theorem, when the martingale is closed by a functional of the process solving the SDE associated to the Kolmogorov equation. In [3, 4, 5] these ideas are furtherly developed and generalized. In [3] the functional Itô's formula is proved for a large class of finite-dimensional càdlàg processes, including semimartingales and Dirichlet processes, and

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for functionals which can depend on the quadratic variation. In [5] the notion of vertical derivative is extended to square integrable continuous martingales and it is proved that it coincides with the stochastic integrand in the martingale representation theorem.

Functional Itô calculus in finite dimension can be also viewed as an application to the spaces of continuous/càdlàg functions of stochastic calculus in Banach spaces ([9, 10, 11, 12, 16]). In [12] the notion of χ -quadratic variation is introduced for Banach space-valued processes (not necessarily semimartingales) and the related Itô's formula is discussed. This general framework finds application to “window” processes in $C([-T, 0], \mathbb{R}^n)$, whose values, at each time $t \in [0, T]$, is essentially the path up to time t of an \mathbb{R}^n -valued continuous process. When applied to window processes, such Itô's formula allows to derive a Clark-Ocone type representation formula by recurring to solutions of a path-dependent Kolmogorov equation. In [16] finite dimensional Itô processes X with constant diffusion coefficient and path-dependent drift are considered. By embedding the dynamics of X into a Banach space of functions $[-T, 0] \rightarrow \mathbb{R}^n$, it is proved that the Feynman-Kac formula provides a solution to the path-dependent backward Kolmogorov equation associated to X , with a non-path-dependent terminal value.

Another approach to path-dependent functionals and path dependent stochastic systems is represented by the embedding in infinite dimensional Hilbert spaces. Indeed, when the dependence on the history is sufficiently regular — precisely regular with respect to a L^2 norm — a representation in the Hilbert space of the form $\mathbb{R} \times L^2$ is possible. This approach goes back to [2] and was further developed in other papers ([14, 15, 19]). With this approach, the very well-developed theory of stochastic calculus in Hilbert space ([8]) can be applied. On the other hand, it leaves out some important classes of problems, in particular all those where the dependence on the history involves pointwise evaluations at past times.

Up to our knowledge, so far the functional Itô calculus has been developed only in finite dimensional spaces. We generalize it to infinite dimension as follows. Consider two real separable Hilbert spaces U, H and a U -valued cylindrical Wiener process W . Given $T > 0$, denote by \mathbb{W} the space $C([0, T], H)$ of continuous functions $[0, T] \rightarrow H$. Given $t \in [0, T]$ and $\mathbf{x} \in \mathbb{W}$, consider the process

$$X_s^{t, \mathbf{x}} = \mathbf{x}_{t \wedge \cdot} + \int_t^{t \vee s} b_r dr + \int_t^{t \vee s} \Phi_r dW_r \quad s \in [0, T],$$

where

$$\mathbf{x}_{t \wedge \cdot}(s) := \begin{cases} \mathbf{x}(s) & s \in [0, t] \\ \mathbf{x}(t) & s \in (t, T], \end{cases}$$

b is a square-integrable H -valued process, and Φ is a square-integrable process valued in the space of Hilbert-Schmidt operators $L_2(U, H)$. We develop a functional Itô calculus for processes of the form

$$u(\cdot, X^{t, \mathbf{x}}) := \left\{ u(s, X_s^{t, \mathbf{x}}) \right\}_{s \in [0, T]}$$

where $u : [0, T] \times \mathbb{W} \rightarrow \mathbb{R}$ is a *non-anticipative* functional, meaning that $u(s, \mathbf{y}) = u(s, \mathbf{y}')$ whenever $\mathbf{y} = \mathbf{y}'$ on $[0, s]$ for a given $s \in [0, T]$. Under suitable regularity assumptions on u , we prove an Itô formula for $u(\cdot, X^{t, \mathbf{x}})$. Then, assuming that $X^{t, \mathbf{x}}$ is driven by an SDE of the form

$$\begin{cases} dX_s = b(s, X)ds + \Phi(s, X)dW_s & \forall s \in [t, T] \\ X_{t \wedge \cdot} = \mathbf{x}_{t \wedge \cdot}, \end{cases} \quad (1.1)$$

where $b: [0, T] \times \mathbb{W} \rightarrow H$, $\Phi: [0, T] \times \mathbb{W} \rightarrow L_2(U, H)$ are non-anticipative coefficients satisfying usual Lipschitz conditions, and letting $f: \mathbb{W} \rightarrow \mathbb{R}$ be a function, we show that, if the non-anticipative function φ defined by

$$\varphi(t, \mathbf{x}) := \mathbb{E}[f(X^{t, \mathbf{x}})] \quad (t, \mathbf{x}) \in [0, T] \times \mathbb{W}$$

is suitably regular, then φ solves the path-dependent backward Kolmogorov equation associated to (1.1) with terminal value f at time T . As a corollary, we obtain a Clark-Ocone type formula for the process $\varphi(\cdot, X^{t, \mathbf{x}})$. Finally, we accomplish a complete study of the regularity of the solution $X^{t, \mathbf{x}}$ to SDE (1.1) with respect to t, \mathbf{x} , when Φ is constant and b contains a convolution of the path of X with a Radon measure. In particular, the case of pointwise delay in the coefficient b will be covered. For the latter class of dynamics, by a pathwise analysis, we show in detail that the assumptions required by the general results previously obtained (Itô's formula, representation of solution to the path-dependent Kolmogorov equation, Clark-Ocone type formula) are satisfied, hence the theory can be applied.

Our methods deviate from the ones used in the aforementioned literature. In [3, 4, 5, 13] non-anticipative functionals are considered on the metric space Λ of couples “(time t , càdlàg path on $[0, t]$)”. Due to the lack of a linear structure for Λ , this choice leads to introduce non-standard notions of derivatives (vertical/horizontal) and to deal with ad-hoc continuity assumptions. On the contrary, we do not use the space Λ and, in a more standard perspective, we look at the set of continuous non-anticipative functionals as a subvector space of the space of continuous functions on $[0, T] \times \mathbb{W}$. Our choice is equivalent to take the restriction of Λ to couples with continuous path in the second component as working space, but shows the advantage to allow to deal with classical Gâteaux derivatives in space. The choice of Gâteaux derivatives in space reveals to be particularly adequate when proving regularity of solutions to path-dependent SDEs with respect to the initial value by using contraction methods in Banach spaces, as in Section 5: if one wishes to apply the theoretical results in practice, this is a key step in order to show that the assumptions of the theory are satisfied. Nevertheless, also in our setting, the introduction of an ad-hoc time derivative for non-anticipative functionals cannot be avoided. It is remarkable that it is convenient for us to use a left-sided time derivative, instead of the right-sided derivative introduced in [13] and then adopted also in [3, 4, 5]. Our choice turns out to be very natural when studying the link between the path-dependent SDE and the associated Kolmogorov equation. Moreover, unlike [9, 10, 11, 12, 16], we do not rephrase our path-dependent problem in a Banach space. This allows to avoid to work with stochastic calculus in Banach spaces.

The paper is organized as follows. In Section 2, after introducing some notation, we define the locally convex space with respect to which the regularity of non-anticipative functionals will be considered. In Section 3 we prove the path-dependent Itô's formula (Theorem 3.8). In Section 4 we show that the Feynman-Kac formula for the strong solution of a path-dependent SDE in Hilbert spaces, if it is enough regular, provides a solution to the associated Kolmogorov equation (Theorem 4.2). We then use this fact to derive a Clark-Ocone type formula (Corollary 4.3). Finally, in Section 5, we explicitly show that the previously developed theory can be applied to a class of SDEs with path-dependent drift and constant diffusion coefficient (Theorem 5.9).

2 Preliminaries

2.1 Notation

Let $T > 0$, let $(\Omega, \mathbb{F} := \{\mathcal{F}_t\}_{t \in [0, T]}, \mathcal{F}, \mathbb{P})$ be a complete filtered probability space, and let $(E, |\cdot|_E)$ be a Banach space. Unless otherwise specified, every Banach space E is considered endowed with its Borel σ -algebra \mathcal{B}_E . $B_b([0, T], E)$ denotes the space of bounded Borel measurable functions $\mathbf{x}: [0, T] \rightarrow E$. If $\mathbf{x} \in B_b([0, T], E)$, then \mathbf{x}_t and $\mathbf{x}(t)$ denote the evaluation at time $t \in [0, T]$ of the function \mathbf{x} , whereas $\mathbf{x}_{t \wedge \cdot}$ denotes the function defined by $(\mathbf{x}_{t \wedge \cdot})_s := \mathbf{x}_{t \wedge s}$ for $s \in [0, T]$. We denote by $B_{b,0}([0, T], E)$ the subspace of $B_b([0, T], E)$ of bounded Borel functions $\mathbf{x}: [0, T] \rightarrow E$ with separable range. Unless otherwise specified, $B_{b,0}([0, T], E)$ is considered with the topology of the uniform convergence. Then $B_{b,0}([0, T], E)$ is a Banach space and $C([0, T], E) \subset B_{b,0}([0, T], E)$. $\mathcal{L}_{\mathcal{P}_T}^0(C([0, T], E))$ denotes the space of E -valued \mathbb{F} -adapted continuous processes. Notice that this implies the measurability of the continuous process

$$\Omega \rightarrow C([0, T], E), \omega \mapsto X(\omega)$$

for all $\omega \in \Omega$ ⁽¹⁾, hence the measurability of

$$(\Omega_T, \mathcal{P}_T) \rightarrow C([0, T], E), (\omega, t) \mapsto X_{t \wedge \cdot}(\omega).$$

If $X, X' \in \mathcal{L}_{\mathcal{P}_T}^0(C([0, T], E))$, we write $X = X'$ if and only if $\mathbb{P}(|X - X'|_\infty = 0) = 1$. For $p \in [1, \infty)$, $\mathcal{L}_{\mathcal{P}_T}^p(C([0, T], E))$ denotes the space of functions $X \in \mathcal{L}_{\mathcal{P}_T}^0(C([0, T], E))$ such that $\Omega \rightarrow C([0, T], E), \omega \mapsto X(\omega)$ has separable range and

$$|X|_{\mathcal{L}_{\mathcal{P}_T}^p(C([0, T], E))} := (\mathbb{E}[|X|_\infty^p])^{1/p} < \infty.$$

By $M([0, T])$ we denote the space of Radon measures on the interval $[0, T]$. For $\nu \in M([0, T])$, $|\nu|_1$ denotes the total variation of ν .

Let F be another Banach space. $\mathcal{G}^n(E, F)$ denotes the space of continuous functions $f: E \rightarrow F$ which are Gâteaux differentiable up to order n and such that, for $j = 1, \dots, n$,

$$E^{i+1} \rightarrow F, (x, y_1, \dots, y_i) \mapsto \partial_{y_1 \dots y_i}^i f(x)$$

is continuous. If $f \in [0, T] \times E \rightarrow F$ is such that $f(t, \cdot) \in \mathcal{G}^n(E, F)$ for all $t \in [0, T]$, then we denote by $\partial_E^j f$, $j = 1, \dots, n$, the Gâteaux differentials of f with respect to E . Similarly, if $f(t, \cdot) \in C^n(E, F)$, i.e. $f(t, \cdot)$ is continuously Fréchet differentiable up to order n , we denote by $D_E^j f$, $j = 1, \dots, n$, the Fréchet differentials of f with respect to E .

$NA([0, T] \times C([0, T], E), F)$ denotes the subspace of $F^{[0, T] \times C([0, T], E)}$ whose members are non-anticipative functions, that is

$$NA([0, T] \times C([0, T], E), F) := \left\{ f \in F^{[0, T] \times C([0, T], E)} : \right. \\ \left. f(t, \mathbf{x}) = f(t, \mathbf{x}_{t \wedge \cdot}) \quad \forall (t, \mathbf{x}) \in [0, T] \times C([0, T], E) \right\}.$$

¹ E is not assumed to be separable.

By $CNA([0, T] \times C([0, T], E), F)$ we denote the subspace of $C([0, T] \times C([0, T], E), F)$ whose members are non-anticipative functions, that is

$$CNA([0, T] \times C([0, T], E), F) := C([0, T] \times C([0, T], E), F) \cap NA([0, T] \times C([0, T], E), F).$$

$(H, |\cdot|_H)$ and $(U, |\cdot|_U)$ denote two real separable Hilbert spaces, with scalar product denoted by $\langle \cdot, \cdot \rangle_H$ and $\langle \cdot, \cdot \rangle_U$, respectively. Let $\epsilon := \{e_n\}_{n \in \mathcal{N}}$ be an orthonormal basis of H , where $\mathcal{N} = \{1, \dots, N\}$ if H has dimension $N \in \mathbb{N} \setminus \{0\}$, or $\mathcal{N} = \mathbb{N}$ if H has infinite dimension. Similarly, $\epsilon' := \{e'_m\}_{m \in \mathcal{M}}$ denotes an orthonormal basis of U , where $\mathcal{M} = \{1, \dots, M\}$ if U has dimension $M \in \mathbb{N} \setminus \{0\}$, or $\mathcal{M} = \mathbb{N}$ if U has infinite dimension. We use the short notation \mathbb{W} for the space $C([0, T], H)$ of continuous functions $[0, T] \rightarrow H$.

2.2 The space $\mathbb{B}_{\sigma^s}^1(E)$

In this section we introduce a topology with respect to which we will often consider the regularity of the differentials of path-dependent functions in the remaining of the manuscript.

Let E denote a Banach space. We begin by introducing on $B_{b,0}([0, T], E)$ the family of seminorms $\mathbf{p}^s := \{p_v^s\}_{v \in M([0, T])}$ defined by

$$p_v^s(\mathbf{x}) := \left| \int_{[0, T]} \mathbf{x}(s) v(ds) \right|_E \quad \forall \mathbf{x} \in B_{b,0}([0, T], E), \quad \forall v \in M([0, T]).$$

Since we are considering only bounded Borel functions \mathbf{x} with separable range, the integral $\int_{[0, T]} \mathbf{x} d\mu$ is well defined.

We denote by σ^s the locally convex vector topology induced on $B_{b,0}([0, T], E)$ by \mathbf{p}^s . If τ_∞ denotes the topology of the uniform convergence $B_{b,0}([0, T], E)$, it is easily seen that

$$\sigma^s \subsetneq \tau_\infty. \quad (2.1)$$

The inclusion $\sigma^s \subset \tau_\infty$ is immediate, whereas the strict inclusion is due to the fact that σ^s is contained in the weak topology of $(B_{b,0}([0, T], E), |\cdot|_\infty)$, and, unless E is trivial, the weak topology is strictly weaker than the topology induced by the norm, because $B_{b,0}([0, T], E)$ is infinite dimensional. The same holds for the restrictions to $C([0, T], E)$, i.e. $\sigma_{|C([0, T], E)}^s \subsetneq \tau_{\infty|C([0, T], E)}$.

Proposition 2.1. *Convergent and Cauchy sequences in σ^s are characterized as follows.*

(i) *A sequence $\{\mathbf{x}_n\}_{n \in \mathbb{N}}$ converges to \mathbf{x} in $(B_{b,0}([0, T], E), \sigma^s)$ if and only if*

$$\begin{cases} (a) & \sup_{n \in \mathbb{N}} |\mathbf{x}_n|_\infty < \infty \\ (b) & \lim_{n \rightarrow \infty} \mathbf{x}_n(s) = \mathbf{x}(s) \quad \forall s \in [0, T]. \end{cases} \quad (2.2)$$

(ii) *A sequence $\{\mathbf{x}_n\}_{n \in \mathbb{N}}$ is Cauchy in $(B_{b,0}([0, T], E), \sigma^s)$ if and only if (2.2)(a) holds and the sequence $\{\mathbf{x}_n(s)\}_{n \in \mathbb{N}}$ is Cauchy for every $s \in [0, T]$.*

Proof. We prove only (i). The proof of (ii) is similar. Suppose that $\{\mathbf{x}_n\}_{n \in \mathbb{N}}$ converges to \mathbf{x} in $(B_{b,0}([0, T], E), \sigma^s)$. For $s \in [0, T]$, if δ_s is the Dirac measure in s , we have

$$\lim_{n \rightarrow \infty} |\mathbf{x}_n(s) - \mathbf{x}(s)|_H = \lim_{n \rightarrow \infty} p_{\delta_s}(\mathbf{x}_n - \mathbf{x}) = 0, \quad ,$$

which shows (2.2)(b).

To show (2.2)(a), consider the family of continuous linear operators

$$\Phi_n : M([0, T]) \rightarrow E, \quad v \mapsto \int_{[0, T]} \mathbf{x}_n(s) v(ds),$$

for $n \in \mathbb{N}$. Since $\{\mathbf{x}_n\}_{n \in \mathbb{N}}$ is convergent, the orbit $\{\Phi_n(v)\}_{n \in \mathbb{N}}$ is bounded, for all $v \in M([0, T])$, then, by Banach-Steinhaus theorem, we have

$$\sup_{n \in \mathbb{N}} |\mathbf{x}_n|_\infty = \sup_{n \in \mathbb{N}} \sup_{\substack{v \in M([0, T]) \\ |v|_1 \leq 1}} \left| \int_{[0, T]} \mathbf{x}_n(s) v(ds) \right|_E = \sup_{n \in \mathbb{N}} |\Phi_n|_{L(M([0, T]), E)} < \infty,$$

where $|\Phi_n|_{L(M([0, T]), E)}$ denotes the operator norm of Φ_n . This shows (2.2)(a) and concludes the proof for one direction of the claim.

Conversely, if (2.2) holds, then $p_v(\mathbf{x}_n - \mathbf{x}) \rightarrow 0$ by Lebesgue's dominated convergence theorem, for all $v \in M([0, T])$, hence $\mathbf{x}_n \rightarrow \mathbf{x}$ in σ^s . \blacksquare

By (2.1), it follows that bounded sets in τ_∞ are bounded in σ^s . By using Banach-Steinhaus theorem similarly as done in the proof of Proposition 2.1, one can see that bounded sets in σ^s are bounded in τ_∞ . Then the bounded sets in σ^s and τ_∞ are the same.

Definition 2.2. We define $\mathbb{B}^1(E)$ as the vector space of all functions $\mathbf{x} : [0, T] \rightarrow E$ which are pointwise limit of a uniformly bounded sequence $\{\mathbf{x}_n\}_{n \in \mathbb{N}} \subset C([0, T], E)$, i.e.

$$\mathbb{B}^1(E) := \left\{ \mathbf{x} \in E^{[0, T]} : \exists \{\mathbf{x}_n\}_{n \in \mathbb{N}} \subset C([0, T], E) \text{ s.t. } \begin{cases} \lim_{n \rightarrow \infty} \mathbf{x}_n(s) = \mathbf{x}(s) \quad \forall s \in [0, T] \\ \sup_{n \in \mathbb{N}} |\mathbf{x}_n|_\infty < \infty \end{cases} \right\}.$$

We denote by $\mathbb{B}_{\sigma^s}^1(E)$ the space $\mathbb{B}^1(E)$ endowed with the locally convex topology induced by σ^s . Then a net $\{\mathbf{x}_i\}_{i \in \mathcal{I}}$ converges to 0 in $\mathbb{B}_{\sigma^s}^1(E)$ if and only if $\lim_i p_v(\mathbf{x}_i) = 0$ for all $v \in M([0, T])$.

Remark 2.3. By Proposition 2.1(i), it follows that $\mathbb{B}^1(E)$ is the sequential closure of $C([0, T], E)$ in $(B_{b,0}([0, T], E), \sigma^s)$. In particular, for any T_2 -space \mathcal{T} and any function $C([0, T], E) \rightarrow \mathcal{T}$, there exists at most one sequentially continuous extension $(\mathbb{B}^1(E), \sigma^s) \rightarrow \mathcal{T}$.

Remark 2.4. In Definition 2.2, by multiplying \mathbf{x}_n by $|\mathbf{x}|_\infty / |\mathbf{x}_n|_\infty$ if necessary, we can assume without loss of generality that $\sup_{n \in \mathbb{N}} |\mathbf{x}_n|_\infty \leq |\mathbf{x}|_\infty$. By Proposition 2.1(i), we then see that the unit ball of $(C([0, T], E), |\cdot|_\infty)$ is σ^s -sequentially dense in the unit ball of $(\mathbb{B}^1(E), |\cdot|_\infty)$.

Since we have the inclusion $\mathbb{B}^1(\mathbb{R}) \subsetneq B_b([0, T], \mathbb{R})$ (see [21, Theorem 11.4]), through the identification $B_b([0, T], \mathbb{R}) = B_b([0, T], \mathbb{R}e)$ in $B_{b,0}([0, T], E)$, for some $e \in E$, $|e|_E = 1$ ($E \neq \{0\}$), we also have the strict inclusion $\mathbb{B}^1(E) \subsetneq B_{b,0}([0, T], E)$.

The space $\mathbb{B}^1(E)$ is closed in $B_{b,0}([0, T], E)$ (hence in $B_b([0, T], E)$) with respect to the uniform norm. The proof of [21, Theorem 11.7], that is made for the case $E = \mathbb{R}$ and for a space of Borel functions larger than our $\mathbb{B}^1(\mathbb{R})$, can be adapted to cover our case. Since the completeness of $\mathbb{B}^1(E)$ is essential to us, we prove it.

Proposition 2.5. $(\mathbb{B}^1(E), |\cdot|_\infty)$ is a Banach space.

Proof. We show that every absolutely convergent sum is convergent in $\mathbb{B}^1(E)$. To this end, let $\{\mathbf{x}_n\}_{n \in \mathbb{N}} \subset \mathbb{B}^1(E)$ be sequence such that $\sum_{n \in \mathbb{N}} |\mathbf{x}_n|_\infty < \infty$. By completeness of $B_b([0, T], E)$, $\sum_{n \in \mathbb{N}} \mathbf{x}_n$ is convergent in $B_b([0, T], E)$, say to \mathbf{z} . We are done if we show that $\mathbf{z} \in \mathbb{B}^1(E)$. By definition of $\mathbb{B}^1(E)$, for each $n \in \mathbb{N}$, there exists a sequence $\{\mathbf{y}_n^{(k)}\}_{k \in \mathbb{N}} \subset C([0, T], E)$ such that

$$M_n := \sup_{k \in \mathbb{N}} |\mathbf{y}_n^{(k)}|_\infty < \infty \quad \text{and} \quad \lim_{k \rightarrow \infty} \mathbf{y}_n^{(k)}(s) = \mathbf{x}_n(s) \quad \forall s \in [0, T].$$

By multiplying $\mathbf{y}_n^{(k)}$ by $|\mathbf{x}_n|_\infty / |\mathbf{y}_n^{(k)}|_\infty$ if necessary, without loss of generality we can assume that $M_n \leq |\mathbf{x}_n|_\infty$. Define $\mathbf{z}_k := \sum_{n=1}^k \mathbf{y}_n^{(k)}$, $k \in \mathbb{N}$. Then $\mathbf{z}_k \in C([0, T], E)$ and

$$\sup_{k \in \mathbb{N}} |\mathbf{z}_k|_\infty \leq \sup_{k \in \mathbb{N}} \sum_{n=1}^k |\mathbf{y}_n^{(k)}|_\infty \leq \sum_{n=1}^\infty |\mathbf{x}_n|_\infty < \infty. \quad (2.3)$$

Moreover, for $s \in [0, T]$, $0 \leq \bar{k} \leq k$,

$$\begin{aligned} |\mathbf{z}(s) - \mathbf{z}_k(s)|_E &= \left| \sum_{n=1}^\infty \mathbf{x}_n(s) - \sum_{n=1}^k \mathbf{y}_n^{(k)}(s) \right|_E \leq \sum_{n=\bar{k}}^\infty (|\mathbf{x}_n|_\infty + |\mathbf{y}_n^{(k)}|_\infty) + \sum_{n=1}^{\bar{k}} |\mathbf{x}_n(s) - \mathbf{y}_n^{(k)}(s)|_E \\ &\leq 2 \sum_{n=\bar{k}}^\infty |\mathbf{x}_n|_\infty + \sum_{n=1}^{\bar{k}} |\mathbf{x}_n(s) - \mathbf{y}_n^{(k)}(s)|_E. \end{aligned}$$

By taking first the $\limsup_{k \rightarrow \infty}$, recalling the pointwise convergence $\mathbf{y}_n^{(k)}(s) \rightarrow \mathbf{x}_n(s)$ as $k \rightarrow \infty$, and then taking the $\lim_{\bar{k} \rightarrow \infty}$, we obtain $\mathbf{z}_k(s) \rightarrow \mathbf{z}(s)$ as $k \rightarrow \infty$. Since $s \in [0, T]$ was arbitrary, this, together with (2.3), proves that $\mathbf{z} \in \mathbb{B}^1(E)$. \blacksquare

2.3 $\mathbb{V}_{\sigma^s}(E)$ -sequentially continuous derivatives

We introduce the following subspace of $\mathbb{B}^1(E)$:

$$\mathbb{V}(E) := \text{Span} \{ \mathbf{x} + v \mathbf{1}_{[t, T]} : \mathbf{x} \in C([0, T], E), v \in E, t \in [0, T] \}. \quad (2.4)$$

A member of $\mathbb{V}(E)$ is the sum of a continuous function and a right-continuous step function (with finite number of jumps). We denote by $\mathbb{V}_{\sigma^s}(E)$ the space $\mathbb{V}(E)$ endowed with the locally convex topology induced by $\mathbb{B}_{\sigma^s}^1(E)$ and by $\mathbb{V}_\infty(E)$ the space $\mathbb{V}(E)$ endowed with the topology induced by the supremum norm $|\cdot|_\infty$.

Definition 2.6. We say that a function $f \in \mathcal{G}^2(C([0, T], E), F)$ has derivatives with $\mathbb{V}_{\sigma^s}(E)$ -sequentially continuous extensions if

$$\partial f : C([0, T], E) \times C([0, T], E) \rightarrow F, (\mathbf{x}, \mathbf{v}) \mapsto \partial_{\mathbf{v}} f(\mathbf{x})$$

and

$$\partial^2 f : C([0, T], E) \times C([0, T], E) \times C([0, T], E) \rightarrow F, (\mathbf{x}, \mathbf{v}, \mathbf{w}) \mapsto \partial_{\mathbf{vw}}^2 f(\mathbf{x})$$

admit sequentially continuous extensions, respectively,

$$\overline{\partial f} : C([0, T], E) \times \mathbb{V}_{\sigma^s}(E) \rightarrow F, (\mathbf{x}, \mathbf{v}) \mapsto \overline{\partial f}(\mathbf{x}).\mathbf{v}$$

and

$$\overline{\partial^2 f} : C([0, T], E) \times \mathbb{V}_{\sigma^s}(E) \times \mathbb{V}_{\sigma^s}(E) \rightarrow F, (\mathbf{x}, \mathbf{v}, \mathbf{w}) \mapsto \overline{\partial^2 f}(\mathbf{x}).(\mathbf{v}, \mathbf{w}).$$

We denote by $\mathcal{G}_{\sigma^s}^2(C([0, T], E), F)$ the subspace of $\mathcal{G}^2(C([0, T], E), F)$ containing the functions having derivatives with $\mathbb{V}_{\sigma^s}(E)$ -sequentially continuous extensions.

If $u \in NA([0, T] \times C([0, T], E), F)$, $t \in [0, T]$, and $u(t, \cdot) \in \mathcal{G}_{\sigma^s}^2(C([0, T], E), F)$, then the notation $\overline{\partial_E u}(t, \mathbf{x}).\mathbf{v}$, for $\mathbf{x} \in C([0, T], E)$ and $\mathbf{v} \in \mathbb{V}(E)$, stands for $\overline{\partial_E u}(t, \cdot)(\mathbf{x}).\mathbf{v}$. Similarly, $\overline{\partial_E u}(t, \cdot)$ stands for $\overline{\partial_E u}(t, \cdot)$.

Remark 2.7. If $u \in NA([0, T] \times C([0, T], E), F)$ is such that, for some $t \in [0, T]$, $u(t, \cdot) \in \mathcal{G}^2(E, F)$, then, by non-anticipativity,

$$\partial_E u(t, \mathbf{x}).\mathbf{v} = \partial_E u(t, \mathbf{x}).\mathbf{v}' \quad \forall \mathbf{x}, \mathbf{v}, \mathbf{v}' \in C([0, T], E) \text{ s.t. } \mathbf{v}(s) = \mathbf{v}'(s) \text{ for } s \in [0, t].$$

If $u(t, \cdot) \in \mathcal{G}_{\sigma^s}^2(C([0, T], E), F)$, then it also holds

$$\overline{\partial_E u}(t, \mathbf{x}).\mathbf{v} = \overline{\partial_E u}(t, \mathbf{x}).\mathbf{v}' \quad \forall \mathbf{x} \in C([0, T], E), \forall \mathbf{v}, \mathbf{v}' \in \mathbb{V}(E) \text{ s.t. } \mathbf{v}(s) = \mathbf{v}'(s) \text{ for } s \in [0, t].$$

In particular,

$$\overline{\partial_E u}(t, \mathbf{x}).(\mathbf{1}_{[t, T]} v) = \overline{\partial_E u}(t, \mathbf{x}).(\mathbf{1}_{[t, T']} v) \quad \forall \mathbf{x} \in C([0, T], E), \forall v \in E, \forall T' \in (t, T).$$

A similar remark holds for the second-order differential. Because of that, the directional derivatives $\overline{\partial_E u}(t, \mathbf{x}).(\mathbf{1}_{[t, T]} v), \overline{\partial_E^2 u}(t, \mathbf{x}).(\mathbf{1}_{[t, T]} v, \mathbf{1}_{[t, T]} w)$, $\mathbf{x} \in C([0, T], E)$, $v, w \in E$, express in our framework the so-called vertical derivatives of [3, 4, 5].

Example 2.8. Let $\mu \in M([0, T])$ and $g \in C([0, T] \times E, F)$ such that $g(t, \cdot) \in \mathcal{G}^2(E, F)$ for all $t \in [0, T]$, and let us assume that $\partial_E g$ and $\partial_E^2 g$ are bounded on bounded sets of $[0, T] \times E$. Define

$$f(\mathbf{x}) := \int_{[0, T]} g(s, \mathbf{x}(s)) \mu(ds) \quad \forall \mathbf{x} \in C([0, T], E).$$

Then $f \in \mathcal{G}^2(C([0, T], E), F)$, with

$$\begin{aligned} \partial f(\mathbf{x}).\mathbf{v} &= \int_{[0, T]} \partial_E g(s, \mathbf{x}(s)).\mathbf{v}(s) \mu(ds) & \forall \mathbf{x}, \mathbf{v} \in C([0, T], H) \\ \partial^2 f(\mathbf{x}).(\mathbf{v}, \mathbf{w}) &= \int_{[0, T]} \partial_E^2 g(s, \mathbf{x}(s)).(\mathbf{v}(s), \mathbf{w}(s)) \mu(ds) & \forall \mathbf{x}, \mathbf{v}, \mathbf{w} \in C([0, T], H). \end{aligned}$$

It is clear that $\partial f(\mathbf{x}).\mathbf{v}$ and $\partial^2 f(\mathbf{x}).(\mathbf{v}, \mathbf{w})$ can be computed with the same expressions when $\mathbf{v}, \mathbf{w} \in \mathbb{V}(E)$. Moreover, by Proposition 2.1(i), by Lebesgue's dominated convergence theorem, and by strong continuity of the Gâteaux differentials of g , we have that $\partial f(\mathbf{x}).\mathbf{v}$ and $\partial f(\mathbf{x}).(\mathbf{v}, \mathbf{w})$ are sequentially continuous with respect to $(\mathbf{x}, \mathbf{v}) \in C([0, T], E) \times \mathbb{V}_{\sigma^s}(E)$ and $(\mathbf{x}, \mathbf{v}, \mathbf{w}) \in C([0, T], E) \times \mathbb{V}_{\sigma^s}(E) \times \mathbb{V}_{\sigma^s}(E)$, respectively. Then $f \in \mathcal{G}_{\sigma^s}^2(C([0, T], E), F)$.

3 A path-dependent Itô's formula

In this section we prove an Itô's formula for processes of the form $\{u(t, X)\}_{t \in [0, T]}$, where X is a diffusion with values in H and u is a non-anticipative function with regular time-space derivatives, in a sense specified below by Assumption 3.3.

For a non-anticipative function u , we introduce the following left-sided time derivative.

Definition 3.1. For $u \in NA([0, T] \times C([0, T], E), F)$ and $(t, \mathbf{x}) \in (0, T) \times C([0, T], E)$, we define the following left-sided derivative, if it exists:

$$\mathcal{D}_t^- u(t, \mathbf{x}) := \lim_{h \rightarrow 0^+} \frac{u(t, \mathbf{x}_{(t-h) \wedge \cdot}) - u(t-h, \mathbf{x})}{h}. \quad (3.1)$$

Remark 3.2. Notice that, by the very definition, for $t, t' \in (0, T)$, $t < t'$, and $\mathbf{x} \in C([0, T], E)$, the derivative $\mathcal{D}_t^- u(t', \mathbf{x}_{t \wedge \cdot})$ coincides with the left-sided derivative of the map

$$(t, T) \rightarrow F, \quad s \mapsto u(s, \mathbf{x}_{t \wedge \cdot})$$

computed in t' .

We will prove the path-dependent Itô's formula under the following assumption.

Assumption 3.3. The function u belongs to $CNA([0, T] \times \mathbb{W}, \mathbb{R})$ and has the following properties.

(i) For all $t \in (0, T)$, $\mathcal{D}_t^- u(t, \mathbf{x})$ exists for all $\mathbf{x} \in \mathbb{W}$. For a.e. $t \in (0, T)$, the map

$$\mathbb{W} \rightarrow \mathbb{R}, \quad \mathbf{x} \mapsto \mathcal{D}_t^- u(t, \mathbf{x})$$

is continuous. For all compact set $K \subset \mathbb{W}$ there exists $M_K > 0$ such that

$$\sup_{\mathbf{x} \in K} |\mathcal{D}_t^- u(t, \mathbf{x})| \leq M_K \quad \text{for a.e. } t \in (0, T). \quad (3.2)$$

(ii) For all $t \in [0, T]$, $u(t, \cdot) \in \mathcal{G}_{\sigma^s}^2(\mathbb{W}, \mathbb{R})$ and the differentials $\partial_{\mathbb{W}} u$ and $\partial_{\mathbb{W}}^2 u$ are bounded:

$$\sup_{t \in [0, T]} \sup_{\substack{\mathbf{x}, \mathbf{v} \in \mathbb{W} \\ |\mathbf{v}|_{\infty} \leq 1}} |\partial_{\mathbb{W}} u(t, \mathbf{x}) \cdot \mathbf{v}| < \infty \quad (3.3)$$

$$\sup_{t \in [0, T]} \sup_{\substack{\mathbf{x}, \mathbf{v}, \mathbf{w} \in \mathbb{W} \\ |\mathbf{w}|_{\infty} |\mathbf{v}|_{\infty} \leq 1}} |\partial_{\mathbb{W}}^2 u(t, \mathbf{x}) \cdot (\mathbf{v}, \mathbf{w})| < \infty. \quad (3.4)$$

(iii) For a.e. $t \in (0, T)$,

$$\lim_{h \rightarrow 0^+} \overline{\partial_{\mathbb{W}} u}(t+h, \mathbf{x}_{t \wedge \cdot}) \cdot (\mathbf{1}_{[t, T]}(\cdot) v) = \overline{\partial_{\mathbb{W}} u}(t, \mathbf{x}_{t \wedge \cdot}) \cdot (\mathbf{1}_{[t, T]}(\cdot) v), \quad (3.5)$$

$$\lim_{h \rightarrow 0^+} \overline{\partial_{\mathbb{W}}^2 u}(t+h, \mathbf{x}_{t \wedge \cdot}) \cdot (\mathbf{1}_{[t, T]}(\cdot) v, \mathbf{1}_{[t, T]}(\cdot) v) = \overline{\partial_{\mathbb{W}}^2 u}(t, \mathbf{x}_{t \wedge \cdot}) \cdot (\mathbf{1}_{[t, T]}(\cdot) v, \mathbf{1}_{[t, T]}(\cdot) v), \quad (3.6)$$

for all $\mathbf{x} \in \mathbb{W}$ and all $v \in H$.

We give some simple examples for which Assumption 3.3 is verified.

Example 3.4. Let $\hat{u} \in C_b^{1,2}([0, T] \times H, \mathbb{R})$ and $u(t, \mathbf{x}) := \hat{u}(t, \mathbf{x}(t))$, $(t, \mathbf{x}) \in [0, T] \times \mathbb{W}$. Then Assumption 3.3 is verified, with $\mathcal{D}_t^- u(t, \mathbf{x}) = \partial_t \hat{u}(t, \mathbf{x}(t))$, for $t \in (0, T)$ and $\mathbf{x} \in \mathbb{W}$, $\overline{\partial_{\mathbb{W}} u}(t, \mathbf{x}) \cdot \mathbf{v} = D_H \hat{u}(t, \mathbf{x}(t)) \cdot \mathbf{v}(t)$, $\overline{\partial_{\mathbb{W}}^2 u}(t, \mathbf{x}) \cdot (\mathbf{v}, \mathbf{w}) = D_H^2 \hat{u}(t, \mathbf{x}(t)) \cdot (\mathbf{v}(t), \mathbf{w}(t))$, for $t \in [0, T]$, $\mathbf{x} \in \mathbb{W}$, $\mathbf{v}, \mathbf{w} \in \mathbb{V}(H)$.

Example 3.5. Let $\gamma \in C^1([0, T], \mathbb{R})$, $h \in C_b^{0,2}([0, T] \times H, \mathbb{R})$. For $(t, \mathbf{x}) \in [0, T] \times \mathbb{W}$, define

$$u(t, \mathbf{x}) := \int_0^t h(s, \mathbf{x}(s)) \gamma(t-s) ds.$$

A direct computation gives, for $(t, \mathbf{x}) \in [0, T] \times \mathbb{W}$,

$$\begin{aligned} \mathcal{D}_t^- u(t, \mathbf{x}) &= h(t, \mathbf{x}(t)) \gamma(0) + \int_0^t h(s, \mathbf{x}(s)) \gamma'(t-s) ds \\ \overline{\partial_{\mathbb{W}} u}(t, \mathbf{x}) \cdot \mathbf{v} &= \int_0^t D_H h(s, \mathbf{x}(s)) \cdot \mathbf{v}(s) \gamma(t-s) ds \\ \overline{\partial_{\mathbb{W}}^2 u}(t, \mathbf{x}) \cdot (\mathbf{v}, \mathbf{w}) &= \int_0^t D_H^2 h(s, \mathbf{x}(s)) \cdot (\mathbf{v}(s), \mathbf{w}(s)) \gamma(t-s) ds \end{aligned}$$

and one can easily see that Assumption 3.3 is verified by u .

Example 3.6. Let u be a function verifying Assumption 3.3 and let $h \in C_b^{1,2}([0, T] \times \mathbb{R}, \mathbb{R})$. For $(t, \mathbf{x}) \in [0, T] \times \mathbb{W}$, define $\hat{u}(t, \mathbf{x}) := h(t, u(t, \mathbf{x}))$. We have

$$\mathcal{D}_t^- \hat{u}(t, \mathbf{x}) = \partial_t h(t, u(t, \mathbf{x})) + D_H u(t, u(t, \mathbf{x})) \cdot \mathcal{D}_t^- u(t, \mathbf{x})$$

and $\overline{\partial_{\mathbb{W}} \hat{u}}, \overline{\partial_{\mathbb{W}}^2 \hat{u}}$ are given by the chain rule. Assumption 3.3 are verified.

Let $B: \mathbb{V}_{\infty}(H) \times \mathbb{V}_{\infty}(H) \rightarrow \mathbb{R}$ be a continuous bilinear functional and let $C > 0$ such that $|B(\mathbf{x}, \mathbf{y})| \leq C|\mathbf{x}|_{\infty}|\mathbf{y}|_{\infty}$, for all $\mathbf{x}, \mathbf{y} \in \mathbb{V}_{\infty}(H)$. Let $\mathbf{a} \in \mathbb{V}(\mathbb{R})$, $|\mathbf{a}|_{\infty} \leq 1$, and $T \in L_2(U, H)$. Then $\mathbf{a}Tu \in \mathbb{V}(H)$, for all $u \in U$, and $\mathbf{a}v \in \mathbb{V}(H)$, for all $v \in H$. Clearly

$$U \times H \rightarrow \mathbb{R}, (u, v) \mapsto B(\mathbf{a}Tu, \mathbf{a}v)$$

is bilinear and continuous. Let $Q \in L(U, H)$ be the unique linear and continuous operator such that

$$\langle Qu, v \rangle_H = B(\mathbf{a}Tu, \mathbf{a}v) \quad \forall u \in U, \forall v \in H. \quad (3.7)$$

We claim that $Q \in L_2(U, H)$. Indeed,

$$\sum_{m \in \mathcal{M}} |Qe'_m|_H^2 = \sum_{m \in \mathcal{M}} \sup_{\substack{v \in H \\ |v|_H \leq 1}} (B(\mathbf{a}Te'_m, \mathbf{a}v))^2 \leq \sum_{m \in \mathcal{M}} C^2 |\mathbf{a}Te'_m|_{\infty}^2 \leq C^2 |T|_{L_2(U, H)}^2 < \infty.$$

Then $Q^* \in L_2(H, U)$ and, by [8, Proposition C.4], $Q^*T \in L(U)$ is a nuclear operator. In particular, the number

$$\sum_{m \in \mathcal{M}} B(\mathbf{a}Te'_m, \mathbf{a}Te'_m) = \sum_{m \in \mathcal{M}} \langle Qe'_m, Te'_m \rangle_H = \sum_{m \in \mathcal{M}} \langle e'_m, Q^*Te'_m \rangle_U = \text{Tr}(Q^*T)$$

is well-defined, finite, and does not depend on the chosen orthonormal basis $\{e'_m\}_{m \in \mathcal{M}}$. This observation leads to introduce the following well-defined notion.

Definition 3.7. Let $B: \mathbb{V}_\infty(H) \times \mathbb{V}_\infty(H) \rightarrow \mathbb{R}$ be a continuous bilinear functional, $\mathbf{a} \in \mathbb{V}(\mathbb{R})$, $T \in L_2(U, H)$. We define

$$\mathbf{T}[B, \mathbf{a}T] := \sum_{m \in \mathcal{M}} B(\mathbf{a}T e'_m, \mathbf{a}T e'_m). \quad (3.8)$$

Let $b \in \mathcal{L}^1_{\mathcal{P}_T}(\mathbb{W})$, $\Phi \in \mathcal{L}^2_{\mathcal{P}_T}(C([0, T], L_2(U, H)))$, and let W be a U -valued cylindrical Wiener process. For $(\hat{t}, \hat{Y}) \in [0, T] \times \mathcal{L}^1_{\mathcal{P}_T}(\mathbb{W})$, let $X^{\hat{t}, \hat{Y}} \in \mathcal{L}^1_{\mathcal{P}_T}(\mathbb{W})$ be the process defined by

$$X_t = \hat{Y}_{\hat{t} \wedge t} + \int_{\hat{t}}^{\hat{t} \vee t} b_s ds + \int_{\hat{t}}^{\hat{t} \vee t} \Phi_s dW_s \quad \forall t \in [0, T]. \quad (3.9)$$

The first main result of the paper is the following path-dependent Itô's formula.

Theorem 3.8. Suppose that u satisfies Assumption 3.3. For $\hat{Y} \in \mathcal{L}^1_{\mathcal{P}_T}(\mathbb{W})$ and $\hat{t} \in [0, T]$, let $X^{\hat{t}, \hat{Y}}$ be the process defined by (3.9). Then

- (i) for all $\omega \in \Omega$, $\mathcal{D}_t^- u(\cdot, X^{\hat{t}, \hat{Y}}(\omega)) \in L^1((0, T), \mathbb{R})$;
- (ii) $\left\{ \overline{\partial_{\mathbb{W}} u}(t, X^{\hat{t}, \hat{Y}}) \cdot (\mathbf{1}_{[t, T]} b_t) \right\}_{t \in [0, T]} \in L^1_{\mathcal{P}_T}(\mathbb{R})$;
- (iii) $\left\{ \overline{\partial_{\mathbb{W}} u}(t, X^{\hat{t}, \hat{Y}}) \cdot (\mathbf{1}_{[t, T]} \Phi_t) \right\}_{t \in [0, T]} \in L^2_{\mathcal{P}_T}(U^*)$;
- (iv) $\left\{ \mathbf{T} \left[\overline{\partial_{\mathbb{W}}^2 u}(t, X^{\hat{t}, \hat{Y}}), \mathbf{1}_{[t, T]} \Phi_t \right] \right\}_{t \in [0, T]} \in L^1_{\mathcal{P}_T}(\mathbb{R})$.

For $t \in [\hat{t}, T]$,

$$\begin{aligned} u(t, X^{\hat{t}, \hat{Y}}) &= u(\hat{t}, \hat{Y}) + \int_{\hat{t}}^t \left(\mathcal{D}_s^- u(s, X^{\hat{t}, \hat{Y}}) ds + \overline{\partial_{\mathbb{W}} u}(s, X^{\hat{t}, \hat{Y}}) \cdot (\mathbf{1}_{[s, T]} b_s) \right) ds \\ &\quad + \frac{1}{2} \int_{\hat{t}}^t \mathbf{T} \left[\overline{\partial_{\mathbb{W}}^2 u}(s, X^{\hat{t}, \hat{Y}}), \mathbf{1}_{[s, T]} \Phi_s \right] ds + \int_{\hat{t}}^t \overline{\partial_{\mathbb{W}} u}(s, X^{\hat{t}, \hat{Y}}) \cdot (\mathbf{1}_{[s, T]} \Phi_s) dW_s, \quad \mathbb{P}\text{-a.e.} \end{aligned} \quad (3.10)$$

Remark 3.9. Notice that, by Example 3.4, (3.10) is a generalization of the standard Itô's formula in the non-path-dependent case.

The proof of Theorem 3.8 is obtained through several partial results. We begin by preparing a setting useful to approximate path-dependent functionals by non-path-dependent ones, for which we can use the standard (non-path-dependent) stochastic analysis on Hilbert spaces, as presented e.g. in [8].

For $n \geq 1$, we consider the product Hilbert space H^n endowed with the scalar product $\langle \cdot, \cdot \rangle_{H^n}$ defined by

$$\langle x, x' \rangle_{H^n} := \sum_{k=1}^n \langle x_k, x'_k \rangle_H \quad \forall x = (x_1, \dots, x_n), x' = (x'_1, \dots, x'_n) \in H^n.$$

Let $\pi := \{0 = t_1 < t_2 < \dots < t_n = T\}$ be a partition of the interval $[0, T]$ and let

$$\delta(\pi) := \sup_{i=1, \dots, n-1} |t_{i+1} - t_i|.$$

Define the operator

$$\ell_\pi: H^n \rightarrow \mathbb{W}$$

as the linear interpolation on the partition π , i.e.

$$\ell_\pi(x_1, \dots, x_n)(t) := x_1 + \sum_{i=1}^{n-1} \frac{t \wedge t_{k+1} - t \wedge t_k}{t_{k+1} - t_k} (x_{k+1} - x_k) \quad \forall t \in [0, T]. \quad (3.11)$$

The operator ℓ_π is linear and continuous, with operator norm 1. If $\mathbf{x} \in \mathbb{W}$ and if $w_{\mathbf{x}}$ denotes a modulus of continuity for \mathbf{x} , then

$$|\ell_\pi(\mathbf{x}_{t_2 \wedge \cdot}(t), \mathbf{x}_{t_3 \wedge \cdot}(t), \dots, \mathbf{x}_{t_{n-1} \wedge \cdot}(t), \mathbf{x}_{t_n \wedge \cdot}(t), \mathbf{x}_{t_n \wedge \cdot}(t)) - \mathbf{x}_{t \wedge \cdot}|_\infty \leq 2w_{\mathbf{x}}(\delta(\pi)). \quad (3.12)$$

Let X be given by (3.9). We introduce the following H -valued processes, obtained by stopping X at certain fixed times. For $i = 1, \dots, n-1$ and $t \in [0, T]$, let $X_t^{(\pi, i)}$ be the continuous process defined by

$$X_t^{(\pi, i)} := X_{t_{i+1} \wedge t}^{\hat{Y}} = \hat{Y}_{\hat{t} \wedge t \wedge t_{i+1}} + \int_{\hat{t}}^{\hat{t} \vee t} \mathbf{1}_{[0, t_{i+1})}(s) b_s ds + \int_{\hat{t}}^{\hat{t} \vee t} \mathbf{1}_{[0, t_{i+1})}(s) \Phi_s dW_s \quad (3.13)$$

and let $X_t^{(\pi, n)} := X_t^{\hat{Y}}$, $t \in [0, T]$. We define the H^n -valued process $X^{(\pi)}$ by

$$X_t^{(\pi)} := (X_t^{(\pi, 1)}, \dots, X_t^{(\pi, n)}) \quad \forall t \in [0, T].$$

Notice that $X^{(\pi)} \in \mathcal{L}_{\mathcal{P}_T}^1(C([0, T], H^n))$. The dynamics of $X^{(\pi)}$ is given by

$$X_t^{(\pi)} = X_{\hat{t}}^{(\pi)} + \int_{\hat{t}}^t b_s^{(\pi)} ds + \int_{\hat{t}}^t \Phi_s^{(\pi)} dW_s \quad \forall t \in [\hat{t}, T],$$

where

$$X_{\hat{t}}^{(\pi)} = (\hat{Y}_{\hat{t} \wedge t_2}, \hat{Y}_{\hat{t} \wedge t_3}, \dots, \hat{Y}_{\hat{t}}, \hat{Y}_{\hat{t}}) \in H^n$$

and where the coefficients $b^{(\pi)}$ and $\Phi^{(\pi)}$ are the following

$$\begin{cases} b_s^{(\pi)} := (\mathbf{1}_{[0, t_2)}(s) b_s, \mathbf{1}_{[0, t_3)}(s) b_s, \dots, \\ \quad \dots, \mathbf{1}_{[0, t_{n-1})}(s) b_s, \mathbf{1}_{[0, t_n)}(s) b_s, \mathbf{1}_{[0, t_n)}(s) b_s) & \forall s \in [0, T] \\ \Phi_s^{(\pi)} u := (\mathbf{1}_{[0, t_2)}(s) \Phi_s u, \mathbf{1}_{[0, t_3)}(s) \Phi_s u, \dots, \\ \quad \dots, \mathbf{1}_{[0, t_{n-1})}(s) \Phi_s u, \mathbf{1}_{[0, t_n)}(s) \Phi_s u, \mathbf{1}_{[0, t_n)}(s) \Phi_s u) & \forall s \in [0, T], \forall u \in U. \end{cases} \quad (3.14)$$

We can verify that $b^{(\pi)} \in L_{\mathcal{P}_T}^1(H^n)$ by

$$\mathbb{E} \left[\int_0^T |b_s^{(\pi)}|_{H^n}^2 ds \right] = \mathbb{E} \left[\int_0^T |b_s|_H^2 \left(1 + \sum_{j=2}^n \mathbf{1}_{[0, t_j)}(s) \right) ds \right] \leq n \mathbb{E} \left[\int_0^T |b_s|_H^2 ds \right]$$

and that $\Phi^{(\pi)} \in L_{\mathcal{P}_T}^2(L_2(U, H^n))$ by

$$\mathbb{E} \left[\int_0^T |\Phi_s^{(\pi)}|_{L_2(U, H^n)}^2 ds \right] = \mathbb{E} \left[\int_0^T |\Phi_s|_{L_2(U, H)}^2 \left(1 + \sum_{j=2}^n \mathbf{1}_{[0, t_j)}(s) \right) ds \right] \leq n \mathbb{E} \left[\int_0^T |\Phi_s|_{L_2(U, H)}^2 ds \right].$$

We notice that, by (3.12) and (3.13),

$$\lim_{\delta \rightarrow 0^+} \sup_{\pi: \delta(\pi) \leq \delta} \sup_{t \in [0, T]} \left| \ell_\pi(X_t^{(\pi)}(\omega)) - X_{t \wedge \cdot}^{\hat{Y}}(\omega) \right|_\infty = 0 \quad \forall \omega \in \Omega. \quad (3.15)$$

Remark 3.10. The importance of the choice of $b^{(\pi)}$ as in (3.14) can be understood when we consider the composition $\ell_\pi(b_s^{(\pi)}(\omega))$. If $\delta(\pi) \rightarrow 0$, then $\ell_\pi(b_s^{(\pi)}(\omega))$ converges pointwise to $\mathbf{1}_{[s,T]}(\cdot)b_s(\omega)$ everywhere on $[0, T]$. On the contrary, if we consider

$$\tilde{b}_s^{(\pi)} := (\mathbf{1}_{[0,t_1)}(s)b_s, \mathbf{1}_{[0,t_2)}(s)b_s, \dots, \mathbf{1}_{[0,t_{n-1})}(s)b_s, \mathbf{1}_{[0,t_n]}(s)b_s)$$

then the pointwise limit as $\delta(\pi) \rightarrow 0$ of $\ell_\pi(\tilde{b}_s^{(\pi)}(\omega))$ is 0 on $[0, s)$ and b_s on $(s, T]$, but it is not guaranteed that the limit in s exists. In our approximation framework, we deal with sequential continuity with respect to the topology σ^s in $\mathbb{V}(H)$, which implies pointwise convergence, as clarified by Proposition 2.1(i). Because of that, the choice of $b^{(\pi)}$ as in (3.14) will be relevant. The same comment holds for $\Phi^{(\pi)}$.

We will need the following measurability lemma.

Lemma 3.11. *Let V, Y, Z be H -valued continuous \mathbb{F} -adapted processes. Let E be a Banach space and let*

$$\bar{f}: \mathbb{W} \times \mathbb{V}_{\sigma^s}(H) \times \mathbb{V}_{\sigma^s}(H) \rightarrow E$$

be a sequentially continuous function. Then the process

$$\Psi := \{\bar{f}(V_{t \wedge \cdot}, \mathbf{1}_{[t,T]}Y_t, \mathbf{1}_{[t,T]}Z_t)\}_{t \in [0,T]}$$

is \mathbb{F} -adapted and left-continuous.

Proof. For all $\mathbf{x} \in \mathbb{W}$, the map

$$[0, T] \rightarrow \mathbb{W}, t \mapsto \mathbf{x}_{t \wedge \cdot}$$

is continuous. Then $\{V_{t \wedge \cdot}\}_{t \in [0,T]}$ is a \mathbb{W} -valued continuous process. We now show that $\{V_{t \wedge \cdot}\}_{t \in [0,T]}$ is \mathbb{F} -adapted. Let $t \in [0, T]$. Let $\pi = \{0 = t_1 < \dots < t_n = T\}$ be a partition of $[0, T]$. It is clear that $(V_{t_1 \wedge t}, \dots, V_{t_n \wedge t})$ is an H^n -valued \mathcal{F}_t -measurable random variable. Then $\ell_\pi(V_{t_1 \wedge t}, \dots, V_{t_n \wedge t})$ is a \mathbb{W} -valued \mathcal{F}_t -adapted random variable. For all $\mathbf{x} \in \mathbb{W}$,

$$|\ell_\pi(\mathbf{x}_{t_1 \wedge t}, \dots, \mathbf{x}_{t_n \wedge t}) - \mathbf{x}_{t \wedge \cdot}|_\infty \leq w_{\mathbf{x}}(\delta(\pi)),$$

where $w_{\mathbf{x}}$ is a modulus of continuity for \mathbf{x} , hence, for all $\omega \in \Omega$,

$$\lim_{\delta(\pi) \rightarrow 0} \ell_\pi(V_{t_1 \wedge t}(\omega), \dots, V_{t_n \wedge t}(\omega)) = V_{t \wedge \cdot}(\omega) \text{ in } \mathbb{W}, \text{ uniformly for } t \in [0, T].$$

This shows that $\{V_{t \wedge \cdot}\}_{t \in [0,T]}$ is a \mathbb{W} -valued \mathbb{F} -adapted process. The same considerations hold for $\{Y_{t \wedge \cdot}\}_{t \in [0,T]}$ and for $\{Z_{t \wedge \cdot}\}_{t \in [0,T]}$.

Now let $t \in [0, T]$ and let $\{\varphi_n\}_{n \in \mathbb{N}} \subset C([0, T], \mathbb{R})$ be a sequence such that

$$\begin{cases} 0 \leq \varphi_n \leq 1 & \forall n \in \mathbb{N} \\ \lim_{n \rightarrow \infty} \varphi_n(s) = \mathbf{1}_{[t,T]}(s) & \forall s \in [0, T]. \end{cases} \quad (3.16)$$

Since, for every $n \in \mathbb{N}$, the map $H \rightarrow \mathbb{W}$, $h \mapsto \varphi_n h$ is linear and continuous, we have that $\varphi_n Y_t$ and $\varphi_n Z_t$ are \mathbb{W} -valued, \mathcal{F}_t -measurable random variables. It follows that $(V_{t \wedge \cdot}, \varphi_n Y_t, \varphi_n Z_t)$ is a $\mathbb{W} \times \mathbb{W} \times \mathbb{W}$ -valued \mathcal{F}_t -measurable random variable. The sequential continuity of \bar{f} implies the continuity of the restriction of \bar{f} to $\mathbb{W} \times \mathbb{W} \times \mathbb{W}$, then

$\bar{f}(V_{t\wedge\cdot}, \varphi_n Y_t, \varphi_n Z_t)$ is an E -valued \mathcal{F}_t -measurable random variable. Now, by (3.16) and Proposition 2.1(i), we have

$$\begin{cases} \lim_{n \rightarrow \infty} \varphi_n Y_t(\omega) = \mathbf{1}_{[t,T]} Y_t(\omega) & \text{in } \mathbb{V}_{\sigma^s}(H), \forall \omega \in \Omega, \\ \lim_{n \rightarrow \infty} \varphi_n Z_t(\omega) = \mathbf{1}_{[t,T]} Z_t(\omega) & \text{in } \mathbb{V}_{\sigma^s}(H), \forall \omega \in \Omega. \end{cases}$$

By sequential continuity of \bar{f} , we conclude

$$\lim_{n \rightarrow \infty} \bar{f}(V_{t\wedge\cdot}, \varphi_n Y_t, \varphi_n Z_t) = \bar{f}(V_{t\wedge\cdot}, \mathbf{1}_{[t,T]} Y_t, \mathbf{1}_{[t,T]} Z_t) \text{ pointwise.}$$

This shows that Ψ_t is an E -valued \mathcal{F}_t -measurable random variable, hence Ψ is \mathbb{F} -adapted.

Let $\{t_n\}_{n \in \mathbb{N}} \subset [0, T]$ be a sequence converging to t in $(0, T]$ from the left. Then the sequence $\{V_{t_n \wedge \cdot}(\omega)\}_{n \in \mathbb{N}}$ converges to $V_{t \wedge \cdot}(\omega)$ in \mathbb{W} , for all $\omega \in \Omega$. Moreover, by Proposition 2.1(i) and continuity of Y, Z ,

$$\forall \omega \in \Omega, \begin{cases} \lim_{n \rightarrow \infty} \mathbf{1}_{[t_n, T]}(\cdot) Y_{t_n}(\omega) = \mathbf{1}_{[t, T]}(\cdot) Y_t & \text{in } \mathbb{V}_{\sigma^s}(H) \\ \lim_{n \rightarrow \infty} \mathbf{1}_{[t_n, T]}(\cdot) Z_{t_n} = \mathbf{1}_{[t, T]}(\cdot) Z_t & \text{in } \mathbb{V}_{\sigma^s}(H). \end{cases}$$

Then, by sequential continuity of \bar{f} , we conclude $\Psi_{t_n}(\omega) \rightarrow \Psi_t(\omega)$. This proves the left continuity of Ψ . \blacksquare

The following proposition provides a version of Itô's formula for Gâteaux differentiable functions that will be used later.

Proposition 3.12. *Let $\tilde{b} \in \mathcal{L}_{\mathcal{P}_T}^1(\mathbb{W})$, $\tilde{\Phi} \in \mathcal{L}_{\mathcal{P}_T}^2(C([0, T], L_2(U, H)))$, and let W be a U -valued cylindrical Wiener process. Let $t_0 \in [0, T]$ and $Y \in \mathcal{L}_{\mathcal{P}_T}^1(\mathbb{W})$. Let $\tilde{X} \in \mathcal{L}_{\mathcal{P}_T}^1(\mathbb{W})$ be the Itô process defined by*

$$\tilde{X}_t = Y_{t \wedge t_0} + \int_{t_0}^{t_0 \vee t} \tilde{b}_s ds + \int_{t_0}^{t_0 \vee t} \tilde{\Phi}_s dW_s \quad \forall t \in [0, T]. \quad (3.17)$$

Let $f: [0, T] \times H \rightarrow \mathbb{R}$ be such that the derivatives $\partial_t f(t, x)$, $\partial_v f(t, x)$, $\partial_{vw}^2 f(t, x)$ exist for all $t \in [0, T]$, $x, v, w \in H$, and are jointly continuous with respect to t, x, v, w . Suppose that

$$\begin{cases} \sup_{(t,x) \in [0,T] \times H} \frac{|\partial_t f(t,x)|}{1 + |x|_H} < \infty \\ \sup_{\substack{(t,x) \in [0,T] \times H \\ v \in H, |v|_H \leq 1}} |\partial_v f(t,x)| < \infty \\ \sup_{\substack{(t,x) \in [0,T] \times H \\ v, w \in H, |v|_H \vee |w|_H \leq 1}} |\partial_{vw}^2 f(t,x)| < \infty. \end{cases} \quad (3.18)$$

Then

- (i) $\{\partial_t f(t, \tilde{X}_t)\}_{t \in [0, T]} \in \mathcal{L}_{\mathcal{P}_T}^1(C([0, T], \mathbb{R}))$;
- (ii) $\{\partial_H f(t, \tilde{X}_t) \cdot \tilde{b}_t\}_{t \in [0, T]} \in L_{\mathcal{P}_T}^1(\mathbb{R})$;

(iii) $\{\partial_H f(t, \tilde{X}_t) \cdot \tilde{\Phi}_t\}_{t \in [0, T]} \in L^2_{\mathcal{P}_T}(U^*);$

(iv) $\{\text{Tr}[\tilde{\Phi}_t^* \partial_H^2 f(t, \tilde{X}_t) \tilde{\Phi}_t]\}_{t \in [0, T]} \in L^1_{\mathcal{P}_T}(\mathbb{R});$

and, for $t \in [t_0, T]$,

$$\begin{aligned} f(t, \tilde{X}_t) = & f(t_0, Y_{t_0}) + \int_{t_0}^t \left(\partial_t f(s, \tilde{X}_s) + \partial_H f(s, \tilde{X}_s) \cdot \tilde{b}_s + \frac{1}{2} \text{Tr}[\tilde{\Phi}_s^* \partial_H^2 f(s, \tilde{X}_s) \tilde{\Phi}_s] \right) ds \\ & + \int_{t_0}^t \partial_H f(s, \tilde{X}_s) \cdot \tilde{\Phi}_s dW_s \quad \mathbb{P}\text{-a.e.} \end{aligned} \quad (3.19)$$

Proof. (i), (ii), (iii), and (iv) are easily obtained by the assumptions on continuity and boundedness of the differentials of f .

We show how to obtain (3.19). Let $\{H_n\}_{n \in \mathbb{N}}$ be an increasing sequence of finite dimensional subspaces of H such that $\bigcup_{n \in \mathbb{N}} H_n$ is dense in H . Let $P_n: H \rightarrow H_n$ be the orthogonal projection of H onto H_n . Define $f_n(t, x) := f(t, P_n x)$ for $(t, x) \in [0, T] \times H$, $n \in \mathbb{N}$. Due to the continuity assumptions on $\partial_t f$, $\partial_H f$, $\partial_H^2 f$, the restriction $f|_{[0, T] \times H_n}$ of f to $[0, T] \times H_n$ belongs to $C^{1,2}([0, T] \times H_n, \mathbb{R})$, hence $f_n \in C^{1,2}([0, T] \times H, \mathbb{R})$. Moreover, (3.18) holds also for f_n , with bounds uniform in n . Then, by [18, p. 69, Theorem 2.10], formula (3.19) holds for all f_n . To conclude the proof it is enough to prove the following limits

$$f_n(t, \tilde{X}_t) \rightarrow f(t, \tilde{X}_t) \quad \mathbb{P}\text{-a.s., } \forall t \in [0, T] \quad (3.20)$$

$$\partial_t f_n(\cdot, \tilde{X} \cdot) \rightarrow \partial_t f(\cdot, \tilde{X} \cdot) \quad \text{in } L^1_{\mathcal{P}_T}(\mathbb{R}) \quad (3.21)$$

$$\partial_H f_n(\cdot, \tilde{X} \cdot) \cdot \tilde{b} \cdot \rightarrow \partial_H f(\cdot, \tilde{X} \cdot) \cdot \tilde{b} \cdot \quad \text{in } L^1_{\mathcal{P}_T}(\mathbb{R}) \quad (3.22)$$

$$\text{Tr}[\tilde{\Phi}^* \partial_H^2 f_n(\cdot, \tilde{X} \cdot) \tilde{\Phi} \cdot] \rightarrow \text{Tr}[\tilde{\Phi}^* \partial_H^2 f(\cdot, \tilde{X} \cdot) \tilde{\Phi} \cdot] \quad \text{in } L^1_{\mathcal{P}_T}(\mathbb{R}) \quad (3.23)$$

$$\partial_H f_n(\cdot, \tilde{X} \cdot) \cdot \tilde{\Phi} \cdot \rightarrow \partial_H f(\cdot, \tilde{X} \cdot) \cdot \tilde{\Phi} \cdot \quad \text{in } L^2_{\mathcal{P}_T}(U^*). \quad (3.24)$$

Convergence (3.20) is clear. Since (3.18) holds with f_n in place of f , with bounds uniform in n , in order to prove (3.21), (3.22), (3.23), (3.24), it is sufficient to show that those convergences hold pointwise. Let $\varphi \in L_2(U, H)$ and $(t, x) \in [0, T] \times H$. Let $\{u_n\}_{n \in \mathbb{N}} \subset U$ be a sequence such that $|u_n|_U \leq 1$ for all n and $u_n \rightarrow u$. Since φ is compact, $\varphi u_n \rightarrow \varphi u$ in H , hence $P_n \varphi u_n \rightarrow \varphi u$. By continuity of $\partial_v f(t, x)$ in t, x, v , we then have

$$\partial_H f_n(t, x) \cdot (\varphi u_n) = \partial_H f(t, P_n x) \cdot (P_n \varphi u_n) \rightarrow \partial_H f(t, x) \cdot (\varphi u).$$

Since we also have $\partial_H f(t, x) \cdot (\varphi u_n) \rightarrow \partial_H f(t, x) \cdot (\varphi u)$, we conclude $\partial_H f_n(t, x) \cdot \varphi \rightarrow \partial_H f(t, x) \cdot \varphi$ in U^* . This provides (3.24). The other pointwise convergences can be proved with similar arguments. \blacksquare

Under the following assumption, we prove in Proposition 3.14 a less general version of Theorem 3.8, in which the functional u is of the form $u(t, \mathbf{x}) = f(\mathbf{x}_{t \wedge \cdot})$.

Assumption 3.13. *The function f belongs to $\mathcal{G}_{\sigma^2}^2(\mathbb{W}, \mathbb{R})$ and its differentials ∂f and $\partial^2 f$ are bounded, that is*

$$M_1 := \sup_{\substack{\mathbf{x}, \mathbf{v} \in \mathbb{W} \\ |\mathbf{v}|_\infty \leq 1}} |\partial f(\mathbf{x}) \cdot \mathbf{v}| < \infty \quad (3.25)$$

$$M_2 := \sup_{\substack{\mathbf{x}, \mathbf{v}, \mathbf{w} \in \mathbb{W} \\ |\mathbf{w}| |\mathbf{v}|_\infty \leq 1}} |\partial^2 f(\mathbf{x}) \cdot (\mathbf{v}, \mathbf{w})| < \infty. \quad (3.26)$$

By Remark 2.4, due to the sequential continuity of the differentials, (3.25) and (3.26) are equivalent to

$$M_1 = \sup_{\substack{\mathbf{x} \in \mathbb{W} \\ \mathbf{v} \in \mathbb{V}(H), |\mathbf{v}|_\infty \leq 1}} \left| \overline{\partial f}(\mathbf{x}).\mathbf{v} \right| < \infty, \quad (3.27)$$

$$M_2 = \sup_{\substack{\mathbf{x} \in \mathbb{W} \\ \mathbf{v}, \mathbf{w} \in \mathbb{V}(H), |\mathbf{w}|_\infty |\mathbf{v}|_\infty \leq 1}} \left| \overline{\partial^2 f}(\mathbf{x}).(\mathbf{v}, \mathbf{w}) \right| < \infty. \quad (3.28)$$

Proposition 3.14. *Suppose that f satisfies Assumption 3.13. For $\hat{Y} \in \mathcal{L}^1_{\mathcal{P}_T}(\mathbb{W})$ and $\hat{t} \in [0, T]$, let $X^{\hat{t}, \hat{Y}}$ be the process defined by (3.9). Then*

- (i) $\left\{ \overline{\partial f}(X^{\hat{t}, \hat{Y}}_{t \wedge \cdot}).(\mathbf{1}_{[t, T]} b_t) \right\}_{t \in [0, T]} \in L^1_{\mathcal{P}_T}(\mathbb{R});$
- (ii) $\left\{ \overline{\partial f}(X^{\hat{t}, \hat{Y}}_{t \wedge \cdot}).(\mathbf{1}_{[t, T]} \Phi_t) \right\}_{t \in [0, T]} \in L^2_{\mathcal{P}_T}(U^*);$
- (iii) $\left\{ \mathbf{T} \left[\overline{\partial^2 f}(X^{\hat{t}, \hat{Y}}_{t \wedge \cdot}), \mathbf{1}_{[t, T]} \Phi_t \right] \right\}_{t \in [0, T]} \in L^1_{\mathcal{P}_T}(\mathbb{R}).$

Moreover, for $t \in [\hat{t}, T]$,

$$\begin{aligned} f(X^{\hat{t}, \hat{Y}}_{t \wedge \cdot}) &= f(\hat{Y}_{\hat{t} \wedge \cdot}) + \int_{\hat{t}}^t \left(\overline{\partial f}(X^{\hat{t}, \hat{Y}}_{s \wedge \cdot}).(\mathbf{1}_{[s, T]} b_s) + \frac{1}{2} \mathbf{T} \left[\overline{\partial^2 f}(X^{\hat{t}, \hat{Y}}_{s \wedge \cdot}), \mathbf{1}_{[s, T]} \Phi_s \right] \right) ds \\ &\quad + \int_{\hat{t}}^t \overline{\partial f}(X^{\hat{t}, \hat{Y}}_{s \wedge \cdot}).(\mathbf{1}_{[s, T]} \Phi_s) dW_s, \quad \mathbb{P}\text{-a.e.} \end{aligned} \quad (3.29)$$

Proof. By Lemma 3.11, the process

$$\left\{ \overline{\partial f}(X^{\hat{t}, \hat{Y}}_{t \wedge \cdot}).(\mathbf{1}_{[t, T]} b_t) \right\}_{t \in [0, T]}$$

is left-continuous and adapted, hence predictable. Similarly, the process

$$\left\{ \overline{\partial f}(X^{\hat{t}, \hat{Y}}_{t \wedge \cdot}).(\mathbf{1}_{[t, T]} \Phi_t u) \right\}_{t \in [0, T]} \quad (3.30)$$

is left-continuous and adapted, hence predictable, for all $u \in U$.

If $(\omega, t) \in \Omega_T$ and $\{u_n\}_{n \in \mathbb{N}}$ is a sequence converging to 0 in U , then

$$\{\mathbf{1}_{[t, T]} \Phi_t(\omega) u_n\}_{n \in \mathbb{N}} \quad (3.31)$$

is a uniformly bounded sequence in $\mathbb{V}(H)$, converging pointwise to 0. Then, by Proposition 2.1(i), the sequence (3.31) converges to 0 in $\mathbb{V}_{\sigma^s}(H)$. By $\mathbb{V}_{\sigma^s}(H)$ -sequential continuity of $\overline{\partial f}(X^{\hat{t}, \hat{Y}}_{t \wedge \cdot}(\omega))$, we conclude

$$\lim_{n \rightarrow \infty} \overline{\partial f}(X^{\hat{t}, \hat{Y}}_{t \wedge \cdot}(\omega)).(\mathbf{1}_{[t, T]} \Phi_t(\omega) u_n) = 0.$$

This shows that, for all $(\omega, t) \in \Omega_T$, $\overline{\partial f}(X^{\hat{t}, \hat{Y}}_{t \wedge \cdot}(\omega)).(\mathbf{1}_{[t, T]} \Phi_t(\omega)) \in U^*$. Then, by separability of U and by Pettis's measurability theorem, we have that

$$\left\{ \overline{\partial f}(X^{\hat{t}, \hat{Y}}_{t \wedge \cdot}).(\mathbf{1}_{[t, T]} \Phi_t) \right\}_{t \in [0, T]}$$

is a U^* -valued predictable process.

We now show the integrability properties in (i) and (ii). By (3.27), we have

$$\mathbb{E} \left[\int_0^T \left| \overline{\partial f}(X_{s\wedge\cdot}^{\hat{t},\hat{Y}}) \cdot (\mathbf{1}_{[s,T]} b_s) \right| ds \right] \leq M_1 T \|b\|_{\mathcal{L}_{\mathcal{P}_T}^1(C([0,T],H))},$$

which concludes the proof of (i). Similarly, by (3.28),

$$\begin{aligned} \mathbb{E} \left[\int_0^T \sup_{\substack{u \in U \\ |u|_U \leq 1}} \left| \overline{\partial f}(X_{s\wedge\cdot}^{\hat{t},\hat{Y}}) \cdot (\mathbf{1}_{[s,T]} \Phi_s u) \right|^2 ds \right] &\leq M_1^2 \mathbb{E} \left[\int_0^T |\Phi_s|_{L(U,H)}^2 ds \right] \\ &\leq M_1^2 T \|\Phi\|_{\mathcal{L}_{\mathcal{P}_T}^2(C([0,T],L_2(U,H)))}^2. \end{aligned}$$

This concludes the proof of (ii).

To show (iii), we first prove that the sum defining $\mathbf{T} \left[\overline{\partial^2 f}(X_{t\wedge\cdot}^{\hat{t},\hat{Y}}), \mathbf{1}_{[t,T]} \Phi_t \right]$ is convergent. By (3.28), we have,

$$\begin{aligned} \sum_{m \in \mathcal{M}} \left| \overline{\partial^2 f}(X_{t\wedge\cdot}^{\hat{t},\hat{Y}}) \cdot (\mathbf{1}_{[t,T]} \Phi_t e'_m, \mathbf{1}_{[t,T]} \Phi_t e'_m) \right| &\leq M_2 \sum_{m \in \mathcal{M}} \|\mathbf{1}_{[t,T]} \Phi_t e'_m\|_\infty^2 \\ &= M_2 \sum_{m \in \mathcal{M}} \|\Phi_t e'_m\|_H^2 \\ &= M_2 \|\Phi_t\|_{L_2(U,H)}^2. \end{aligned} \tag{3.32}$$

Then $\mathbf{T} \left[\overline{\partial^2 f}(X_{t\wedge\cdot}^{\hat{t},\hat{Y}}), \mathbf{1}_{[t,T]} \Phi_t \right]$ is well defined, for all $t \in [0, T]$. By Lemma 3.11, for every $m \in \mathcal{M}$, the process

$$\left\{ \overline{\partial^2 f}(t, X_{t\wedge\cdot}^{\hat{t},\hat{Y}}) \cdot (\mathbf{1}_{[t,T]} \Phi_t e'_m, \mathbf{1}_{[t,T]} \Phi_t e'_m) \right\}_{t \in [0, T]} \tag{3.33}$$

is adapted and left-continuous, hence predictable. Then $\left\{ \mathbf{T} \left[\overline{\partial^2 f}(X_{t\wedge\cdot}^{\hat{t},\hat{Y}}), \mathbf{1}_{[t,T]} \Phi_t \right] \right\}_{t \in [0, T]}$ is predictable. It is also integrable, by (3.32).

We finally address formula (3.29). We will derive it from the standard Itô's formula in Hilbert spaces, by using the approximation framework introduced at pp. 11–12.

Since, by Assumption 3.13, $f \in \mathcal{G}^2(\mathbb{W}, \mathbb{R})$, by linearity of ℓ_π we have that

$$f_\pi: H^n \rightarrow \mathbb{R}, \quad x \mapsto f(\ell_\pi(x))$$

f_π is strongly continuously Gâteaux differentiable up to order 2 on H^n , with

$$\partial f_\pi(x) \cdot v = \partial f(\ell_\pi(x)) \cdot \ell_\pi(v), \tag{3.34}$$

for all $(x, v) \in H^n \times H^n$,

$$\partial^2 f_\pi(x) \cdot (v, w) = \partial^2 f(\ell_\pi(x)) \cdot (\ell_\pi(v), \ell_\pi(w)), \tag{3.35}$$

for all $(x, v, w) \in H^n \times H^n \times H^n$. Then we can apply the standard Itô's formula, in the version provided by Proposition 3.12, to the predictable pathwise continuous process

$$\left\{ f_\pi(X_t^{(\pi)}) \right\}_{t \in [0, T]} = \left\{ f(\ell_\pi(X_t^{(\pi)})) \right\}_{t \in [0, T]}.$$

For $t \in [\hat{t}, T]$, we have

$$\begin{aligned} f_\pi(X_t^{(\pi)}) &= f_\pi(X_{\hat{t}}^{(\pi)}) + \int_{\hat{t}}^t \left(\partial f_\pi(X_s^{(\pi)}) \cdot b_s^{(\pi)} \frac{1}{2} \text{Tr} \left((\Phi_s^{(\pi)})^* \partial^2 f_\pi(X_s^{(\pi)}) \Phi_s^{(\pi)} \right) \right) ds \\ &\quad + \int_{\hat{t}}^t \partial f_\pi(X_s^{(\pi)}) \cdot \Phi_s^{(\pi)} dW_s \quad \mathbb{P}\text{-a.e.} \end{aligned} \quad (3.36)$$

Through several steps, we are going to prove that the terms appearing in (3.36) converge to the corresponding terms in (3.29), as $\delta(\pi) \rightarrow 0$.

Let $\{\pi_n\}_{n \in \mathbb{N}}$ be a sequence of partition of $[0, T]$ such that $\lim_{n \rightarrow \infty} \delta(\pi_n) = 0$.

Step 1. By (3.15) and by continuity of f , we immediately have that, for $t \in [0, T]$, $f_\pi(X_t^{(\pi_n)}) \rightarrow f(X_{t \wedge \cdot}^{\hat{t}, \hat{Y}})$ \mathbb{P} -a.e..

Step 2. We show that

$$\lim_{n \rightarrow \infty} \partial f_{\pi_n}(X_{\#}^{(\pi_n)}) \cdot (b_{\#}^{(\pi_n)}) = \overline{\partial f}(X_{\# \wedge \cdot}^{\hat{t}, \hat{Y}}) \cdot (\mathbf{1}_{[\#, T]} b_{\#}) \text{ in } L^1_{\mathcal{D}_T}(\mathbb{R}). \quad (3.37)$$

We notice that, by the very definition of $b_s^{(\pi_n)}$ in (3.14) and of ℓ_π (see also Remark 3.10), we have, for all $\omega \in \Omega$ and $s \in [0, T]$,

$$\ell_{\pi_n}(b_s^{(\pi_n)}(\omega)) = \begin{cases} b_s(\omega) & \text{on } [s, T] \\ 0 & \text{on } [0, s - 2\delta(\pi_n)] \end{cases}$$

and $\sup_{n \in \mathbb{N}} |\ell_{\pi_n}(b_s^{(\pi_n)}(\omega))|_\infty \leq |b_s(\omega)|_H$. By Proposition 2.1(i), it follows

$$\lim_{n \rightarrow \infty} \ell_{\pi_n}(b_s^{(\pi_n)}(\omega)) = \mathbf{1}_{[s, T]} b_s(\omega) \text{ in } \mathbb{V}_{\sigma^s}(H), \quad \forall (\omega, s) \in \Omega_T. \quad (3.38)$$

By (3.27) and (3.34),

$$\begin{aligned} &\sup_{s \in [0, T]} |\partial f_{\pi_n}(X_s^{(\pi_n)}) \cdot b_s^{(\pi_n)}| + \sup_{s \in [0, T]} |\overline{\partial f}(X_{s \wedge \cdot}^{\hat{t}, \hat{Y}}) \cdot (\mathbf{1}_{[s, T]} b_s)| \\ &\leq M_1 \left(\sup_{s \in [0, T]} |\ell_{\pi_n}(b_s^{(\pi_n)})|_\infty + \sup_{s \in [0, T]} |\mathbf{1}_{[s, T]} b_s|_\infty \right) = 2M_1 |b|_\infty. \end{aligned} \quad (3.39)$$

By (3.15), (3.38), (3.39), sequential continuity of $\overline{\partial f}$, and Lebesgue's dominated convergence theorem, we obtain

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\int_0^T \left| \partial f_{\pi_n}(X_s^{(\pi_n)}) \cdot b_s^{(\pi_n)} - \overline{\partial f}(X_{s \wedge \cdot}^{\hat{t}, \hat{Y}}) \cdot (\mathbf{1}_{[s, T]} b_s) \right| ds \right] = 0,$$

which provides (3.37).

Step 3. We show that

$$\lim_{n \rightarrow \infty} \partial f_{\pi_n}(X_{\#}^{(\pi_n)}) \cdot \Phi_{\#}^{(\pi_n)} = \overline{\partial f}(X_{\# \wedge \cdot}^{\hat{t}, \hat{Y}}) \cdot (\mathbf{1}_{[\#, T]} \Phi_{\#}) \text{ in } L^2_{\mathcal{D}_T}(U^*). \quad (3.40)$$

Let $\{u_n\}_{n \in \mathbb{N}}$ be a sequence weakly convergent to u in the unit ball of U . Since $\Phi_s(\omega)$ is compact, $\Phi_s(\omega)u_n \rightarrow \Phi_s(\omega)u$ strongly in H for all $(\omega, s) \in \Omega_T$. We also have, for $n \in \mathbb{N}$,

$$\ell_{\pi_n}(\Phi_s^{(\pi_n)}(\omega)u_n) = \begin{cases} \Phi_s(\omega)u_n & \text{on } [s, T] \\ 0 & \text{on } [0, s - 2\delta(\pi_n)]. \end{cases}$$

and

$$\sup_{n \in \mathbb{N}} \left| \ell_{\pi_n}(\Phi_s^{(\pi_n)}(\omega)u_n) \right|_{\infty} \leq \sup_{n \in \mathbb{N}} |\Phi_s(\omega)u_n|_H \leq |\Phi_s(\omega)|_{L(U,H)}.$$

Then, by Proposition 2.1(i),

$$\lim_{n \rightarrow \infty} \ell_{\pi_n}(\Phi_s^{(\pi_n)}(\omega)u_n) = \mathbf{1}_{[s,T]} \Phi_s(\omega)u \text{ in } \mathbb{V}_{\sigma^s}(H), \forall (\omega, s) \in \Omega_T. \quad (3.41)$$

By (3.15), (3.34), (3.41), we obtain

$$\lim_{n \rightarrow \infty} \left| \partial f_{\pi_n}(X_s^{(\pi_n)})(\Phi_s^{(\pi_n)}u_n) - \overline{\partial f}(X_{s \wedge \cdot}^{\hat{t}, \hat{Y}})(\mathbf{1}_{[s,T]} \Phi_s u) \right| = 0 \quad \forall (\omega, s) \in \Omega_T.$$

By (3.41) and sequential continuity of $\overline{\partial f}$, we have

$$\lim_{n \rightarrow \infty} \left| \overline{\partial f}(X_{s \wedge \cdot}^{\hat{t}, \hat{Y}})(\mathbf{1}_{[s,T]} \Phi_s(u_n - u)) \right| = 0 \quad \forall (\omega, s) \in \Omega_T.$$

Since the weakly convergent sequence $\{u_n\}_{n \in \mathbb{N}}$ is arbitrary, the two limits above let us to conclude

$$\lim_{n \rightarrow \infty} \left| \partial f_{\pi_n}(X_s^{(\pi_n)})(\Phi_s^{(\pi_n)}u) - \overline{\partial f}(X_{s \wedge \cdot}^{\hat{t}, \hat{Y}})(\mathbf{1}_{[s,T]} \Phi_s u) \right|_{U^*} = 0 \quad \forall (\omega, s) \in \Omega_T. \quad (3.42)$$

Moreover, by (3.27) and (3.34), for $u \in U$, $|u|_U = 1$,

$$\begin{aligned} & \sup_{s \in [0, T]} |\partial f_{\pi_n}(X_s^{(\pi_n)})(\Phi_s^{(\pi_n)}u)| + \sup_{s \in [0, T]} |\overline{\partial f}(X_{s \wedge \cdot}^{\hat{t}, \hat{Y}})(\mathbf{1}_{[s,T]} \Phi_s u)| \\ & \leq M_1 \left(\sup_{s \in [0, T]} \left| \ell_{\pi_n}(\Phi_s^{(\pi_n)}u) \right|_{\infty} + \sup_{s \in [0, T]} |\mathbf{1}_{[s,T]} \Phi_s u|_{\infty} \right) \\ & \leq 2M_1 \sup_{s \in [0, T]} |\Phi_s|_{L(U,H)} \leq 2M_1 \sup_{s \in [0, T]} |\Phi_s|_{L_2(U,H)}. \end{aligned} \quad (3.43)$$

By (3.42), (3.43), and by Lebesgue's dominated convergence theorem, we obtain

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\int_0^T \left| \partial f_{\pi_n}(X_s^{(\pi_n)})(\Phi_s^{(\pi_n)}u) - \overline{\partial f}(X_{s \wedge \cdot}^{\hat{t}, \hat{Y}})(\mathbf{1}_{[s,T]} \Phi_s u) \right|_{U^*}^2 ds \right] = 0.$$

This provides (3.40).

Step 4. We show that

$$\lim_{n \rightarrow \infty} \text{Tr} \left((\Phi_{\#}^{(\pi_n)})^* \partial^2 f_{\pi_n}(X_{\#}^{(\pi_n)}) \Phi_{\#}^{(\pi_n)} \right) = \mathbf{T} \left[\overline{\partial^2 f}(X_{\# \wedge \cdot}^{\hat{t}, \hat{Y}}), \mathbf{1}_{[\#, T]} \Phi_{\#} \right] \text{ in } L^1_{\mathcal{P}_T}(\mathbb{R}). \quad (3.44)$$

By

$$\begin{aligned} & \left| \partial^2 f_{\pi_n}(X_s^{(\pi_n)})(\Phi_s^{(\pi_n)}e'_m, \Phi_s^{(\pi_n)}e'_m) \right| + \left| \overline{\partial^2 f}(X_{s \wedge \cdot}^{\hat{t}, \hat{Y}})(\mathbf{1}_{[s,T]} \Phi_s e'_m, \mathbf{1}_{[s,T]} \Phi_s e'_m) \right| \\ & \leq M_2 \left(|\ell_{\pi_n}(\Phi_s^{(\pi_n)}e'_m)|_{\infty}^2 + |\mathbf{1}_{[s,T]} \Phi_s e'_m|_{\infty}^2 \right) = 2M_2 |\Phi_s e'_m|_H^2, \end{aligned}$$

and

$$\sum_{m \in \mathcal{M}} \mathbb{E} \left[\int_0^T |\Phi_s e'_m|_H^2 ds \right] = \mathbb{E} \left[\int_0^T |\Phi_s|_{L_2(U,H)}^2 ds \right] < \infty,$$

we can apply Lebesgue's dominated convergence theorem and obtain

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \mathbb{E} \left[\int_0^T \left| \text{Tr} \left((\Phi_s^{(\pi_n)})^* \partial^2 f_{\pi_n}(X_s^{(\pi_n)}) \Phi_s^{(\pi_n)} \right) - \mathbf{T} \left[\overline{\partial^2 f}(X_{s \wedge \cdot}), \mathbf{1}_{[s, T]} \Phi_s \right] \right| ds \right] \\
& \leq \lim_{n \rightarrow \infty} \sum_{m \in \mathcal{M}} \mathbb{E} \left[\int_0^T \left| \partial^2 f_{\pi_n}(X_s^{(\pi_n)}), (\Phi_s^{(\pi_n)} e'_m, \Phi_s^{(\pi_n)} e'_m) \right. \right. \\
& \quad \left. \left. - \overline{\partial^2 f}(X_{s \wedge \cdot}^{\hat{i}, \hat{Y}}), (\mathbf{1}_{[s, T]} \Phi_s e'_m, \mathbf{1}_{[s, T]} \Phi_s e'_m) \right| ds \right] \\
& = \sum_{m \in \mathcal{M}} \mathbb{E} \left[\int_0^T \lim_{n \rightarrow \infty} \left| \partial^2 f_{\pi_n}(X_s^{(\pi_n)}), (\Phi_s^{(\pi_n)} e'_m, \Phi_s^{(\pi_n)} e'_m) \right. \right. \\
& \quad \left. \left. - \overline{\partial^2 f}(X_{s \wedge \cdot}^{\hat{i}, \hat{Y}}), (\mathbf{1}_{[s, T]} \Phi_s e'_m, \mathbf{1}_{[s, T]} \Phi_s e'_m) \right| ds \right] \\
& = 0
\end{aligned}$$

where the pointwise convergence of the latter integrand comes from the sequential continuity of $\partial^2 f$, from (3.15), and from

$$\lim_{n \rightarrow \infty} \ell_{\pi_n}(\Phi_s^{(\pi_n)}(\omega) e'_m) = \mathbf{1}_{[s, T]} \Phi_s(\omega) e'_m \text{ in } \mathbb{V}_{\sigma^s}(H), \quad \forall (\omega, s) \in \Omega_T, \quad \forall m \in \mathcal{M}$$

(that comes from (3.41) with $u_n = u = e'_m$ for all n).

Step 5. We can now conclude the proof of the theorem, because (3.29) is obtained by passing to the limit $n \rightarrow \infty$ in (3.36) (with π replaced by π_n), and by considering the partial results of Step 1, Step 2, Step 3, Step 4. \blacksquare

We can now prove Theorem 3.8.

Proof of Theorem 3.8. (i) By continuity of u , for $h \in (0, T)$, both $\{u(t, X_{(t-h) \wedge \cdot}^{\hat{i}, \hat{Y}})\}_{t \in [h, T]}$ and $\{u(t-h, X_{(t-h) \wedge \cdot}^{\hat{i}, \hat{Y}})\}_{t \in [h, T]}$ are pathwise continuous and \mathbb{F} -adapted, hence predictable. In particular, $\mathcal{D}_t^- u(\cdot, X^{\hat{i}, \hat{Y}})$ is predictable on $(0, T)$ and then $\mathcal{D}_t^- u(\cdot, X^{\hat{i}, \hat{Y}}(\omega))$ is measurable for all $\omega \in \Omega$. Moreover, for $\omega \in \Omega$, the map $[0, T] \rightarrow \mathbb{W}$, $t \mapsto X_{t \wedge \cdot}^{\hat{i}, \hat{Y}}(\omega)$ is continuous, hence $\{X_{t \wedge \cdot}^{\hat{i}, \hat{Y}}(\omega)\}_{t \in [0, T]}$ is compact in \mathbb{W} and (3.2) implies $\mathcal{D}_t^- u(\cdot, X^{\hat{i}, \hat{Y}}(\omega)) \in L^1((0, T), \mathbb{R})$.

(ii)+ (iii)+ (iv) For $n \geq 1$, let $t_k^n := kT/n$, for $k = 0, \dots, n$. By applying Lemma 3.11 to $\overline{\partial_{\mathbb{W}} u}(t_k^n, \cdot)$, for $k = 1, \dots, n$, we obtain the predictability of the process

$$\left\{ \overline{\partial_{\mathbb{W}} u}(t_k^n, X_{t \wedge \cdot}^{\hat{i}, \hat{Y}}), (\mathbf{1}_{[t, T]} b_t) \right\}_{t \in [0, T]} \in L^1_{\mathcal{P}_T}(\mathbb{R}) \quad \forall k = 1, \dots, n.$$

By Assumption 3.3(iii), for all $t \in (0, T]$ and all $\omega \in \Omega$,

$$\overline{\partial_{\mathbb{W}} u}(t, X_{t \wedge \cdot}^{\hat{i}, \hat{Y}}(\omega)), (\mathbf{1}_{[t, T]} b_t(\omega)) = \lim_{n \rightarrow \infty} \sum_{k=1}^n \mathbf{1}_{(t_{k-1}^n, t_k^n]}(t) \overline{\partial_{\mathbb{W}} u}(t_k^n, X_{t \wedge \cdot}^{\hat{i}, \hat{Y}}(\omega)), (\mathbf{1}_{[t, T]} b_t(\omega)),$$

which shows that $\left\{ \overline{\partial_{\mathbb{W}} u}(t, X_{t \wedge \cdot}^{\hat{i}, \hat{Y}}), (\mathbf{1}_{[t, T]} b_t) \right\}_{t \in [0, T]}$ is predictable.

In the same way, by applying Lemma 3.11 and Pettis's measurability theorem, we see that the U^* -valued process $\left\{ \overline{\partial_{\mathbb{W}} u}(t, X_{t \wedge \cdot}^{\hat{i}, \hat{Y}}), (\mathbf{1}_{[t, T]} \Phi_t) \right\}_{t \in [0, T]}$ is predictable.

We now address $\left\{ \mathbf{T} \left[\overline{\partial_{\mathbb{W}}^2 u}(t, X^{\hat{t}, \hat{Y}}), \mathbf{1}_{[t, T]} \Phi_t \right] \right\}_{t \in [0, T]}$. Again by Lemma 3.11, the process

$$\left\{ \overline{\partial_{\mathbb{W}}^2 u}(t_k^n, X_{t_{k-1}^n \wedge \cdot}^{\hat{t}, \hat{Y}}) \cdot (\mathbf{1}_{[t, T]} \Phi_t e'_m, \mathbf{1}_{[t, T]} \Phi_t e'_m) \right\}_{t \in [0, T]}$$

is predictable, for all $m \in \mathcal{M}$. Thanks to Assumption 3.3(iii), we have, for all $t \in (0, T]$ and $\omega \in \Omega$,

$$\begin{aligned} \overline{\partial_{\mathbb{W}}^2 u}(t, X_{t_{k-1}^n \wedge \cdot}^{\hat{t}, \hat{Y}}) \cdot (\mathbf{1}_{[t, T]} \Phi_t e'_m, \mathbf{1}_{[t, T]} \Phi_t e'_m) \\ = \lim_{n \rightarrow \infty} \sum_{k=1}^n \mathbf{1}_{(t_{k-1}^n, t_k^n]}(t) \overline{\partial_{\mathbb{W}}^2 u}(t_k^n, X_{t_{k-1}^n \wedge \cdot}^{\hat{t}, \hat{Y}}) \cdot (\mathbf{1}_{[t, T]} \Phi_t e'_m, \mathbf{1}_{[t, T]} \Phi_t e'_m). \end{aligned}$$

Then $\left\{ \overline{\partial_{\mathbb{W}}^2 u}(t, X^{\hat{t}, \hat{Y}}) \cdot (\mathbf{1}_{[t, T]} \Phi_t e'_m, \mathbf{1}_{[t, T]} \Phi_t e'_m) \right\}_{t \in [0, T]}$ is predictable, hence

$$\left\{ \mathbf{T} \left[\overline{\partial_{\mathbb{W}}^2 u}(t, X^{\hat{t}, \hat{Y}}), \mathbf{1}_{[t, T]} \Phi_t \right] \right\}_{t \in [0, T]}$$

is predictable too.

Finally, the integrability properties claimed in (ii), (iii), (iv) are proved exactly as for Proposition 3.14(i), (ii), (iii) by using Assumption 3.3(ii).

We now prove formula (3.10). Considering Remark 2.7, without loss of generality we can assume $t = T$. Let $n \geq 1$ and let $\hat{t} = t_0^n < \dots < t_n^n = T$ be a partition of $[\hat{t}, T]$, with $t_k^n - t_{k-1}^n = (T - \hat{t})/n$, for $k = 1, \dots, n$. We first write

$$\begin{aligned} u(T, X^{\hat{t}, \hat{Y}}) - u(\hat{t}, \hat{Y}) &= \sum_{k=1}^n \left(u(t_k^n, X^{\hat{t}, \hat{Y}}) - u(t_{k-1}^n, X^{\hat{t}, \hat{Y}}) \right) \\ &= \sum_{k=1}^n \left(u(t_k^n, X^{\hat{t}, \hat{Y}}) - u(t_k^n, X_{t_{k-1}^n \wedge \cdot}^{\hat{t}, \hat{Y}}) \right) + \sum_{k=1}^n \left(u(t_k^n, X_{t_{k-1}^n \wedge \cdot}^{\hat{t}, \hat{Y}}) - u(t_{k-1}^n, X^{\hat{t}, \hat{Y}}) \right) \\ &=: \mathbf{I}_n + \mathbf{II}_n. \end{aligned} \quad (3.45)$$

For $k = 1, \dots, n$, due to our assumptions on u , we can apply Proposition 3.14 to $u(t_k^n, \cdot)$, then (3.29) gives

$$\begin{aligned} u(t_k^n, X^{\hat{t}, \hat{Y}}) &= u(t_k^n, \hat{Y}_{\hat{t} \wedge \cdot}) + \int_{\hat{t}}^{t_k^n} \left(\overline{\partial_{\mathbb{W}} u}(t_k^n, X_{s \wedge \cdot}^{\hat{t}, \hat{Y}}) \cdot (\mathbf{1}_{[s, T]} b_s) + \frac{1}{2} \mathbf{T} \left[\overline{\partial_{\mathbb{W}}^2 u}(t_k^n, X_{s \wedge \cdot}^{\hat{t}, \hat{Y}}), \mathbf{1}_{[s, T]} \Phi_s \right] \right) ds \\ &\quad + \int_{\hat{t}}^{t_k^n} \overline{\partial_{\mathbb{W}} u}(t_k^n, X_{s \wedge \cdot}^{\hat{t}, \hat{Y}}) \cdot (\mathbf{1}_{[s, T]} \Phi_s) dW_s \\ &= u(t_k^n, X_{t_{k-1}^n \wedge \cdot}^{\hat{t}, \hat{Y}}) + \int_{t_{k-1}^n}^{t_k^n} \left(\overline{\partial_{\mathbb{W}} u}(t_k^n, X_{s \wedge \cdot}^{\hat{t}, \hat{Y}}) \cdot (\mathbf{1}_{[s, T]} b_s) + \frac{1}{2} \mathbf{T} \left[\overline{\partial_{\mathbb{W}}^2 u}(t_k^n, X_{s \wedge \cdot}^{\hat{t}, \hat{Y}}), \mathbf{1}_{[s, T]} \Phi_s \right] \right) ds \\ &\quad + \int_{t_{k-1}^n}^{t_k^n} \overline{\partial_{\mathbb{W}} u}(t_k^n, X_{s \wedge \cdot}^{\hat{t}, \hat{Y}}) \cdot (\mathbf{1}_{[s, T]} \Phi_s) dW_s, \quad \mathbb{P}\text{-a.e..} \end{aligned}$$

Then

$$\begin{aligned} \mathbf{I}_n &= \int_{\hat{t}}^T \sum_{k=1}^n \mathbf{1}_{(t_{k-1}^n, t_k^n]}(s) \left(\overline{\partial_{\mathbb{W}} u}(t_k^n, X_{s \wedge \cdot}^{\hat{t}, \hat{Y}}) \cdot (\mathbf{1}_{[s, T]} b_s) + \frac{1}{2} \mathbf{T} \left[\overline{\partial_{\mathbb{W}}^2 u}(t_k^n, X_{s \wedge \cdot}^{\hat{t}, \hat{Y}}), \mathbf{1}_{[s, T]} \Phi_s \right] \right) ds \\ &\quad + \int_{\hat{t}}^T \sum_{k=1}^n \mathbf{1}_{(t_{k-1}^n, t_k^n]}(s) \overline{\partial_{\mathbb{W}} u}(t_k^n, X_{s \wedge \cdot}^{\hat{t}, \hat{Y}}) \cdot (\mathbf{1}_{[s, T]} \Phi_s) dW_s, \quad \mathbb{P}\text{-a.e..} \end{aligned}$$

By Assumption 3.3(ii),(iii), we can apply Lebesgue's dominated convergence theorem (the integrands are estimated similarly as done in Steps 2–4 of the proof of Proposition 3.14) and obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} \sum_{k=1}^n \mathbf{1}_{(t_{k-1}^n, t_k^n]}(\#) & \left(\overline{\partial_{\mathbb{W}} u}(t_k^n, X_{\# \wedge \cdot}^{\hat{t}, \hat{Y}}) \cdot (\mathbf{1}_{[\#, T]} b_{\#}) + \frac{1}{2} \mathbf{T} \left[\overline{\partial_{\mathbb{W}}^2 u}(t_k^n, X_{\# \wedge \cdot}^{\hat{t}, \hat{Y}}), \mathbf{1}_{[\#, T]} \Phi_{\#} \right] \right) \\ & = \overline{\partial_{\mathbb{W}} u}(\#, X_{\# \wedge \cdot}^{\hat{t}, \hat{Y}}) \cdot (\mathbf{1}_{[\#, T]} b_{\#}) + \frac{1}{2} \mathbf{T} \left[\overline{\partial_{\mathbb{W}}^2 u}(\#, X_{\# \wedge \cdot}^{\hat{t}, \hat{Y}}), \mathbf{1}_{[\#, T]} \Phi_{\#} \right] \text{ in } L^1_{\mathcal{P}_T}(\mathbb{R}) \end{aligned}$$

and

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \mathbf{1}_{(t_{k-1}^n, t_k^n]}(\#) \overline{\partial_{\mathbb{W}} u}(t_k^n, X_{\# \wedge \cdot}^{\hat{t}, \hat{Y}}) \cdot (\mathbf{1}_{[\#, T]} \Phi_{\#}) = \overline{\partial_{\mathbb{W}} u}(\#, X_{\# \wedge \cdot}^{\hat{t}, \hat{Y}}) \cdot (\mathbf{1}_{[\#, T]} \Phi_{\#}) \text{ in } L^2_{\mathcal{P}_T}(U^*).$$

The two limits above permit to obtain the following limit in $L^1(\Omega, \mathbb{R})$:

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbf{I}_n & = \int_{\hat{t}}^T \left(\overline{\partial_{\mathbb{W}} u}(s, X_{s \wedge \cdot}^{\hat{t}, \hat{Y}}) \cdot (\mathbf{1}_{[s, T]} b_s) + \frac{1}{2} \mathbf{T} \left[\overline{\partial_{\mathbb{W}}^2 u}(s, X_{s \wedge \cdot}^{\hat{t}, \hat{Y}}), \mathbf{1}_{[s, T]} \Phi_s \right] \right) ds \\ & \quad + \int_{\hat{t}}^T \overline{\partial_{\mathbb{W}} u}(s, X_{s \wedge \cdot}^{\hat{t}, \hat{Y}}) \cdot (\mathbf{1}_{[s, T]} \Phi_s) dW_s. \end{aligned} \quad (3.46)$$

We now address the term \mathbf{II}_n . By Assumption 3.3(i), continuity of u , and recalling Remark 3.2, we can apply [17, (1.4.4), p. 23] and conclude that $(t, T) \rightarrow \mathbb{R}$, $t \mapsto u(s, \mathbf{x}_{t \wedge \cdot})$ is Lipschitz. We can then write

$$\begin{aligned} \mathbf{II}_n & = \sum_{k=1}^n \int_{t_{k-1}^n}^{t_k^n} \frac{d}{ds} u(s, X_{t_{k-1}^n \wedge \cdot}^{\hat{t}, \hat{Y}}) ds = \sum_{k=1}^n \int_{t_{k-1}^n}^{t_k^n} \mathcal{D}_t^- u(s, X_{t_{k-1}^n \wedge \cdot}^{\hat{t}, \hat{Y}}) ds \\ & = \int_{\hat{t}}^T \left(\sum_{k=1}^n \mathbf{1}_{(t_{k-1}^n, t_k^n]}(s) \mathcal{D}_t^- u(s, X_{t_{k-1}^n \wedge \cdot}^{\hat{t}, \hat{Y}}) \right) ds. \end{aligned} \quad (3.47)$$

Fix $\omega \in \Omega$. As noticed at the beginning of the proof, the set $K := \{X_{t \wedge \cdot}^{\hat{t}, \hat{Y}}(\omega)\}_{t \in [0, T]}$ is compact in \mathbb{W} . Then, by Assumption 3.3(i), there exists $M_K > 0$ (depending on ω , since our compact set K depends on ω too) such that

$$\left| \sum_{k=1}^n \mathbf{1}_{(t_{k-1}^n, t_k^n]}(s) \mathcal{D}_t^- u(s, X_{t_{k-1}^n \wedge \cdot}^{\hat{t}, \hat{Y}}(\omega)) \right|_H \leq M_K \text{ for a.e. } s \in (0, T). \quad (3.48)$$

For fixed $s \in (0, T)$, let $\{k_n\}_{n \in \mathbb{N}}$ be the sequence such that $s \in (t_{k_n-1}^n, t_{k_n}^n]$ for all $n \in \mathbb{N}$, $n \geq 1$. Then $X_{t_{k_n-1}^n \wedge \cdot}^{\hat{t}, \hat{Y}}(\omega) \rightarrow X_{s \wedge \cdot}^{\hat{t}, \hat{Y}}(\omega)$ in \mathbb{W} as $n \rightarrow \infty$. Since this holds for all $s \in (0, T)$ and since $\mathbb{W} \rightarrow \mathbb{R}$, $\mathbf{x} \mapsto \mathcal{D}_t^- u(s, \mathbf{x})$, is continuous for a.e. $s \in (0, T)$ because of Assumption 3.3(i), we have

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \mathbf{1}_{(t_{k-1}^n, t_k^n]}(s) \mathcal{D}_t^- u(s, X_{t_{k-1}^n \wedge \cdot}^{\hat{t}, \hat{Y}}(\omega)) = \mathcal{D}_t^- u(s, X_{s \wedge \cdot}^{\hat{t}, \hat{Y}}(\omega)) \text{ for a.e. } s \in (0, T). \quad (3.49)$$

By (3.48) and (3.49), we can apply Lebesgue's dominated convergence theorem to (3.47) evaluated in ω and obtain

$$\lim_{n \rightarrow \infty} \mathbf{II}_n(\omega) = \int_{\hat{t}}^T \mathcal{D}_t^- u(s, X_{s \wedge \cdot}^{\hat{t}, \hat{Y}}(\omega)) ds.$$

Since $\omega \in \Omega$ was arbitrary, we have

$$\lim_{n \rightarrow \infty} \Pi_n = \int_{\hat{t}}^T \mathcal{D}_t^- u(s, X_{s \wedge \cdot}^{\hat{t}, \hat{Y}}) ds \text{ pointwise on } \Omega. \quad (3.50)$$

This concludes the proof, because, by passing to the limit $n \rightarrow \infty$ in (3.45) and considering (3.46) and (3.50), we obtain (3.10) with $t = T$. \blacksquare

4 Application to path-dependent PDEs

In this section we use the path-dependent Itô's formula to relate the solution of an H -valued path-dependent SDE with a path-dependent Kolmogorov equation, similarly as in the classical non-path-dependent case (see e.g. [7, Ch. 7]). As a corollary, we will derive a Clark-Ocone type formula.

The following assumption on b, Φ will be standing for the remaining of the present section.

Assumption 4.1. $b \in CNA([0, T] \times \mathbb{W}, H)$, $\Phi \in CNA([0, T] \times \mathbb{W}, L_2(U, H))$, and there exists $M > 0$ such that

$$\begin{cases} |b(t, \mathbf{x}) - b(t, \mathbf{x}')|_H \leq M|\mathbf{x} - \mathbf{x}'|_\infty \\ |b(t, \mathbf{x})|_H \leq M(1 + |\mathbf{x}|_\infty) \end{cases} \quad \begin{cases} |\Phi(t, \mathbf{x}) - \Phi(t, \mathbf{x}')|_{L_2(U, H)} \leq M|\mathbf{x} - \mathbf{x}'|_\infty \\ |\Phi(t, \mathbf{x})|_{L_2(U, H)} \leq M(1 + |\mathbf{x}|_\infty) \end{cases}$$

for all $t \in [0, T]$, $\mathbf{x}, \mathbf{x}' \in \mathbb{W}$.

For $p > 2$, $\hat{Y} \in \mathcal{L}_{\mathcal{P}_T}^p(\mathbb{W})$, $\hat{t} \in [0, T]$, we consider the following path-dependent SDE

$$\begin{cases} dX_s = b(s, X)ds + \Phi(s, X)dW_s & \forall s \in [\hat{t}, T] \\ X_{\hat{t} \wedge \cdot} = \hat{Y}_{\hat{t} \wedge \cdot} \end{cases} \quad (4.1)$$

By a standard contraction argument (see e.g. [18, Ch. 3] and [6, Theorem 3.6]), there exists a unique strong solution $X^{\hat{t}, \hat{Y}}$ to (4.1) in $\mathcal{L}_{\mathcal{P}_T}^p(\mathbb{W})$, i.e. a unique process $X^{\hat{t}, \hat{Y}} \in \mathcal{L}_{\mathcal{P}_T}^p(\mathbb{W})$ such that, for all $t \in [0, T]$,

$$X_t^{\hat{t}, \hat{Y}} = \hat{Y}_{\hat{t} \wedge t} + \int_{\hat{t}}^{\hat{t} \vee t} b(r, X^{\hat{t}, \hat{Y}})dr + \int_{\hat{t}}^{\hat{t} \vee t} \Phi(r, X^{\hat{t}, \hat{Y}})dW_r \quad \mathbb{P}\text{-a.e.}$$

Moreover, the map

$$[0, T] \times \mathcal{L}_{\mathcal{P}_T}^p(\mathbb{W}) \rightarrow \mathcal{L}_{\mathcal{P}_T}^p(\mathbb{W}), (t, Y) \mapsto X^{t, Y} \quad (4.2)$$

is Lipschitz continuous with respect to Y , uniformly for $t \in [0, T]$, and jointly continuous in (t, Y) . Uniqueness of solution yields the flow property

$$X^{t, \mathbf{x}} = X^{s, X^{t, \mathbf{x}}} \text{ in } \mathcal{L}_{\mathcal{P}_T}^p(\mathbb{W}), \forall (t, \mathbf{x}) \in [0, T] \times \mathbb{W}, \forall s \in [t, T]. \quad (4.3)$$

Let $f: \mathbb{W} \rightarrow \mathbb{R}$ be a Lipschitz function. Hereafter in this section, we denote by φ the function

$$\varphi: [0, T] \times \mathbb{W} \rightarrow \mathbb{R}$$

defined by

$$\varphi(t, \mathbf{x}) := \mathbb{E} [f(X^{t, \mathbf{x}})] \quad \forall (t, \mathbf{x}) \in [0, T] \times \mathbb{W}. \quad (4.4)$$

Due to the continuity properties of the map (4.2), $\varphi(t, \mathbf{x})$ is Lipschitz continuous with respect to \mathbf{x} , uniformly for $t \in [0, T]$, and jointly continuous in (t, \mathbf{x}) . It is clear that $\varphi(t, \mathbf{x}) = \varphi(t, \mathbf{x}_{t \wedge \cdot})$. Then $\varphi \in CNA([0, T] \times \mathbb{W}, \mathbb{R})$. Since $X^{t, \mathbf{x}}$ is independent of \mathcal{F}_t , we can write, by (4.3) and [1, Lemma 3.9, p. 55],

$$\begin{aligned} \varphi(t', \mathbf{x}) &= \mathbb{E} [f(X^{t', \mathbf{x}})] = \mathbb{E} [f(X^{t, X_{t \wedge \cdot}^{t', \mathbf{x}}})] \\ &= \mathbb{E} \left[\mathbb{E} [f(X^{t, X_{t \wedge \cdot}^{t', \mathbf{x}}}) | \mathcal{F}_t] \right] = \mathbb{E} [\varphi(t, X_{t \wedge \cdot}^{t', \mathbf{x}})] = \mathbb{E} [\varphi(t, X^{t', \mathbf{x}})] \quad \forall t \in [t', T]. \end{aligned} \quad (4.5)$$

In what follows, we will show that, in case $\varphi(t, \mathbf{x})$ is sufficiently regular with respect to the variable \mathbf{x} , then Proposition 3.14 can be used to conclude that $\mathcal{D}_t^- \varphi$ exists everywhere and that φ solves a path-dependent backward Kolmogorov equation associated to SDE (4.1). We argue similarly as in [7, Ch. 7], where, differently than in our case, the setting is non-path-dependent. The two main tools of the argument are (4.5) and formula (3.29).

In order to use formula (3.29), we need to make some assumptions regarding existence and regularity of the spatial derivatives of φ . In this section, we make such assumptions without any further investigation under which conditions they can be obtained. We only guess that, at least in the Markovian case, i.e. when b and Φ are not path-dependent, and the only path-dependence is due to f , the regularity assumptions on $\varphi(t, \cdot)$ should come from continuity assumption on ∂f and $\partial^2 f$ with respect to σ^s , and from regularity assumptions on the coefficients b and Φ , thanks to the results in [7, Ch. 7]. In the following section, we will prove that the regularity assumptions on the spatial derivatives of φ are satisfied for a particular class of dynamics X .

For a function $v(t, \mathbf{x})$, defined for $(t, \mathbf{x}) \in [0, T] \times \mathbb{B}^1(H)$, the more concise notation $\partial_{\mathbb{B}^1} v$ stands for $\partial_{\mathbb{B}^1(H)} v$, and $\partial_{\mathbb{B}^1}^2 v$ stands for $\partial_{\mathbb{B}^1(H)}^2 v$. For a function v such that, for all $t \in (0, T)$, $v(t, \cdot)$ satisfies Assumption 3.13, we define $\mathcal{L}v$ by

$$\mathcal{L}v(t, \mathbf{x}) := \overline{\partial_{\mathbb{W}} v}(t, \mathbf{x}) \cdot (\mathbf{1}_{[t, T]} b(t, \mathbf{x})) + \frac{1}{2} \mathbf{T} \left[\overline{\partial_{\mathbb{W}}^2 v}(t, \mathbf{x}), \mathbf{1}_{[t, T]} \Phi(t, \mathbf{x}) \right] \quad \forall (t, \mathbf{x}) \in (0, T) \times \mathbb{W}.$$

Theorem 4.2. *Let φ be defined by (4.4). If φ satisfies Assumption 3.3(ii), then φ satisfies also Assumption 3.3(i) and*

$$\mathcal{D}_t^- \varphi(t, \mathbf{x}) + \mathcal{L}\varphi(t, \mathbf{x}) = 0 \quad \forall (t, \mathbf{x}) \in (0, T) \times \mathbb{W}. \quad (4.6)$$

Proof. Let $t', t \in (0, T)$, $t' < t$, $\mathbf{x} \in \mathbb{W}$. By assumption on the spatial derivatives of $\varphi(t, \cdot)$, we can apply Proposition 3.14 to $\varphi(t, X_{t \wedge \cdot}^{t', \mathbf{x}})$, and obtain

$$\begin{aligned} \mathbb{E} [\varphi(t, X_{t \wedge \cdot}^{t', \mathbf{x}})] &= \varphi(t, \mathbf{x}_{t' \wedge \cdot}) + \int_{t'}^t \mathbb{E} \left[\overline{\partial_{\mathbb{W}} \varphi}(t, X_{s \wedge \cdot}^{t', \mathbf{x}}) \cdot (\mathbf{1}_{[s, T]} b(s, X^{t', \mathbf{x}})) \right] ds \\ &\quad + \frac{1}{2} \int_{t'}^t \mathbb{E} \left[\mathbf{T} \left[\overline{\partial_{\mathbb{W}}^2 \varphi}(t, X_{s \wedge \cdot}^{t', \mathbf{x}}), \mathbf{1}_{[s, T]} \Phi(s, X^{t', \mathbf{x}}) \right] \right] ds. \end{aligned} \quad (4.7)$$

By non-anticipativity, $\varphi(t, X^{t', \mathbf{x}}) = \varphi(t, X_{s \wedge \cdot}^{t', \mathbf{x}})$. Then, by (4.5) and (4.7), we have

$$\begin{aligned} \varphi(t, \mathbf{x}_{t' \wedge \cdot}) - \varphi(t', \mathbf{x}) &= - \int_{t'}^t \mathbb{E} \left[\overline{\partial_{\mathbb{W}} \varphi}(t, X_{s \wedge \cdot}^{t', \mathbf{x}}) \cdot (\mathbf{1}_{[s, T]} b(s, X^{t', \mathbf{x}})) \right] ds \\ &\quad - \frac{1}{2} \int_{t'}^t \mathbb{E} \left[\mathbf{T} \left[\overline{\partial_{\mathbb{W}}^2 \varphi}(t, X_{s \wedge \cdot}^{t', \mathbf{x}}), \mathbf{1}_{[s, T]} \Phi(s, X^{t', \mathbf{x}}) \right] \right] ds. \end{aligned}$$

By continuity of (4.2),

$$\lim_{t' \rightarrow t^-} \sup_{s \in [t', t]} |X_{s \wedge \cdot}^{t', \mathbf{x}} - \mathbf{x}_{t \wedge \cdot}|_H = 0 \text{ on } \Omega. \quad (4.8)$$

By non-anticipativity and continuity of b and Φ , we then obtain, on Ω ,

$$\begin{aligned} \lim_{t' \rightarrow t^-} \sup_{s \in [t', t]} |b(s, X^{t', \mathbf{x}}) - b(t, \mathbf{x})|_H &= 0 \\ \lim_{t' \rightarrow t^-} \sup_{s \in [t', t]} |\Phi(s, X^{t', \mathbf{x}}) - \Phi(t, \mathbf{x})|_{L_2(U, H)} &= 0. \end{aligned}$$

Then, by Proposition 2.1(i), for any sequence $\{(t'_n, s_n)\}_{n \in \mathbb{N}}$ with $t'_n \leq s_n \leq t$ and $t'_n \rightarrow t$, we have

$$\begin{cases} \lim_{n \rightarrow \infty} \mathbf{1}_{[s_n, T]} b(s_n, X^{t'_n, \mathbf{x}}) = \mathbf{1}_{[t, T]} b(t, \mathbf{x}) \text{ in } \mathbb{V}_{\sigma^s}(H) \\ \lim_{n \rightarrow \infty} \mathbf{1}_{[s_n, T]} \Phi(s_n, X^{t'_n, \mathbf{x}}) = \mathbf{1}_{[t, T]} \Phi(t, \mathbf{x}) \text{ in } \mathbb{V}_{\sigma^s}(L_2(U, H)). \end{cases} \quad (4.9)$$

By assumption, $\overline{\partial_{\mathbb{W}} \varphi}(t, \mathbf{x}) \cdot \mathbf{v}$ and $\overline{\partial_{\mathbb{W}}^2 \varphi}(t, \mathbf{x}) \cdot (\mathbf{v}, \mathbf{v})$ are uniformly bounded for $\mathbf{x} \in \mathbb{W}$ and $\mathbf{v} \in \mathbb{V}(H)$, $|\mathbf{v}|_\infty \leq 1$, and sequentially continuous in $(\mathbf{x}, \mathbf{v}) \in \mathbb{W} \times \mathbb{V}_{\sigma^s}(H)$. Then, by (4.8), (4.9), and Lebesgue's dominated convergence theorem, we have

$$\lim_{t' \rightarrow t^-} \sup_{s \in [t', t]} \mathbb{E} \left[\left| \overline{\partial_{\mathbb{W}} \varphi}(t, X_{s \wedge \cdot}^{t', \mathbf{x}}) \cdot (\mathbf{1}_{[s, T]} b(s, X^{t', \mathbf{x}})) - \overline{\partial_{\mathbb{W}} \varphi}(t, \mathbf{x}) \cdot (\mathbf{1}_{[t, T]} b(t, \mathbf{x})) \right| \right] = 0 \quad (4.10)$$

and

$$\lim_{t' \rightarrow t^-} \sup_{s \in [t', t]} \mathbb{E} \left[\left| \mathbf{T} \left[\overline{\partial_{\mathbb{W}}^2 \varphi}(t, X_{s \wedge \cdot}^{t', \mathbf{x}}), \mathbf{1}_{[s, T]} \Phi(s, X^{t', \mathbf{x}}) \right] - \mathbf{T} \left[\overline{\partial_{\mathbb{W}}^2 \varphi}(t, \mathbf{x}), \mathbf{1}_{[t, T]} \Phi(t, \mathbf{x}) \right] \right| \right] = 0 \quad (4.11)$$

Thanks to (4.10) and (4.11), we can finally write

$$\lim_{t' \rightarrow t^-} \frac{1}{t - t'} \int_{t'}^t \mathbb{E} \left[\overline{\partial_{\mathbb{W}} \varphi}(t, X_{s \wedge \cdot}^{t', \mathbf{x}}) \cdot (\mathbf{1}_{[s, T]} b(s, X^{t', \mathbf{x}})) \right] ds = \overline{\partial_{\mathbb{W}} \varphi}(t, \mathbf{x}) \cdot (\mathbf{1}_{[t, T]} b(t, \mathbf{x}))$$

and

$$\lim_{t' \rightarrow t^-} \frac{1}{t - t'} \int_{t'}^t \mathbb{E} \left[\mathbf{T} \left[\overline{\partial_{\mathbb{W}}^2 \varphi}(t, X_{s \wedge \cdot}^{t', \mathbf{x}}), \mathbf{1}_{[s, T]} \Phi(s, X^{t', \mathbf{x}}) \right] \right] ds = \mathbf{T} \left[\overline{\partial_{\mathbb{W}}^2 \varphi}(t, \mathbf{x}), \mathbf{1}_{[t, T]} \Phi(t, \mathbf{x}) \right]$$

This proves that $\mathcal{D}_t^- \varphi(t, \mathbf{x})$ exists and that (4.6) holds true.

We now show that $\mathcal{D}_t^- \varphi(t, \mathbf{x})$ is continuous in \mathbf{x} and that

$$\sup_{\substack{t \in (0, T) \\ \mathbf{x} \in K}} |\mathcal{D}_t^- \varphi(t, \mathbf{x})| < \infty,$$

for all compact sets $K \subset \mathbb{W}$. By (4.6), it is sufficient to show that

$$\mathbb{W} \rightarrow \mathbb{R}, \mathbf{x} \mapsto \mathcal{L}\varphi(t, \mathbf{x})$$

is continuous, for all $t \in (0, T)$, and that

$$\sup_{\substack{t \in (0, T) \\ \mathbf{x} \in K}} |\mathcal{L}\varphi(t, \mathbf{x})| < \infty.$$

But this is straightforward from the sublinear growth and continuity assumptions in \mathbf{x} of b, Φ and from the boundedness and continuity assumption on $\overline{\partial_{\mathbb{W}}\varphi}, \overline{\partial_{\mathbb{W}}^2\varphi}$. ■

Corollary 4.3. *If φ satisfies Assumption 3.3(ii),(iii), then, for all $t \in [\hat{t}, T]$, we have the following representation:*

$$\varphi(t, X^{\hat{t}, \hat{Y}}) = \varphi(\hat{t}, \hat{Y}) + \int_{\hat{t}}^t \overline{\partial_{\mathbb{W}}\varphi}(s, X^{\hat{t}, \hat{Y}}) \cdot (\mathbf{1}_{[s, T]} \Phi_s) dW_s \quad \mathbb{P}\text{-a.e.} \quad (4.12)$$

Proof. By Theorem 4.2, the assumptions of Theorem 3.8 are satisfied for φ . By applying formula (3.10) to $\varphi(t, X^{\hat{t}, \hat{Y}})$ and recalling (4.6), we obtain (4.12). ■

5 The case $b(t, \mathbf{x}) = b(t, \int_{[0, T]} \tilde{\mathbf{x}}(t-s) \mu(ds))$ and additive noise

In this section, in a case of interest, we show that Theorem 4.2 and Corollary 4.3 can be applied.

The following assumption will be standing for the remaining of this section.

Assumption 5.1.

(i) $\mu \in M([0, T])$;

(ii) $b: [0, T] \times H \rightarrow H$ is continuous and there exists $N > 0$ such that

$$\sup_{t \in [0, T]} |b(t, y)|_H \leq N(1 + |y|_H) \quad \forall y \in H, \quad (5.1)$$

$$\sup_{t \in [0, T]} |b(t, y) - b(t, y')|_H \leq N|y - y'|_H \quad \forall y, y' \in H. \quad (5.2)$$

(iii) for all $t \in [0, T]$, $b(t, \cdot) \in \mathcal{G}^2(H, H)$,

$$N_1 := \sup_{\substack{(t, y) \in [0, T] \times H \\ v \in H, |v|_H \leq 1}} |\partial_H b(t, y) \cdot v|_H < \infty, \quad (5.3)$$

$$N_2 := \sup_{\substack{(t, y) \in [0, T] \times H \\ v, w \in H, |v|_H \vee |w|_H \leq 1}} |\partial_H^2 b(t, y) \cdot (v, w)|_H < \infty, \quad (5.4)$$

and $\partial_H b(t, y) \cdot v, \partial_H^2 b(t, y) \cdot (v, w)$ are jointly continuous in t, y, v, w .

We define

$$\hat{b}(t, \mathbf{y}) := b\left(t, \int_{[0, T]} \tilde{\mathbf{y}}(t-s) \mu(ds)\right) \quad \forall (t, \mathbf{y}) \in [0, T] \times \mathbb{B}^1(H).$$

where

$$\tilde{\mathbf{y}}(r) := \mathbf{1}_{[-T, 0)}(r) \mathbf{y}(0) + \mathbf{1}_{[0, T]}(r) \mathbf{y}(r) \quad \forall r \in [-T, T]. \quad (5.5)$$

Then $\hat{b}(t, \mathbf{y})$ is a function of t and the convolution between μ and \mathbf{y} computed taking into account the past history of \mathbf{y} on the time window $[t-T, t]$.

Remark 5.2. The fact that $b(t, \cdot) \in \mathcal{G}^2(H, H)$, with differentials uniformly bounded, implies that $b(t, \cdot) \in C_b^1(H, H)$, i.e. $b(t, \cdot)$ is Fréchet differentiable and the Fréchet differential $Db(t, \cdot)$ is continuous and bounded (with bound uniform in t , due to our assumptions on b). For the proof, see [7, Proposition 7.4.1].

Let again W denote a U -valued cylindrical Wiener process and let $B \in L_2(U, H)$. Consider the following SDE:

$$\begin{cases} dX_s = \hat{b}(s, X) ds + B dW_s & \forall s \in [\hat{t}, T] \\ X_{\hat{t} \wedge \cdot} = \hat{Y}_{\hat{t} \wedge \cdot}, \end{cases} \quad (5.6)$$

for $\hat{Y} \in \mathcal{L}_{\mathcal{F}_T}^p(\mathbb{W})$, $p > 2$. Notice that Assumption 4.1 is verified with the present coefficients b and $\Phi \equiv B$. Our aim is to prove a certain regularity of the solution $X^{\hat{t}, \hat{Y}}$ of (5.6) with respect to the initial datum $\hat{Y} \in \mathcal{L}_{\mathcal{F}_T}^p(\mathbb{W})$, $p > 2$, suitable to apply Theorem 4.2 and Corollary 4.3.

Remark 5.3. The choice $\mu = \delta_0$, Dirac measure in 0, corresponds to the Markovian case $\hat{b}(s, \mathbf{y}) = b(t, \mathbf{y}(t))$. By choosing $\mu = \delta_a$, Dirac measure centered in $a \in (0, T]$, we obtain a drift $\hat{b}(t, \mathbf{y}) = b(t, \mathbf{y}(t-a))$ with a pointwise delay.

By Assumption 5.1, we have that $\hat{b}(\cdot, \mathbf{y})$ is continuous for all $\mathbf{y} \in C([0, T], H)$. Moreover,

$$|\hat{b}(t, \mathbf{y}_1) - \hat{b}(t, \mathbf{y}_2)|_H \leq N \int_{[0, T]} |\tilde{\mathbf{y}}_1(t-r) - \tilde{\mathbf{y}}_2(t-r)|_H \mu(dr) \quad \forall \mathbf{y}_1, \mathbf{y}_2 \in \mathbb{B}^1(H).$$

Then, if $\{\mathbf{y}_n\}_{n \in \mathbb{N}} \subset \mathbb{W}$ and $\mathbf{y}_n \rightarrow \mathbf{y}$ in $\mathbb{B}_{\sigma^s}^1(H)$, we have $\hat{b}(t, \mathbf{y}_n) \rightarrow \hat{b}(t, \mathbf{y})$ for all $t \in [0, T]$. Hence $\hat{b}(\cdot, \mathbf{y}) \in \mathbb{B}^1(H)$, for all $\mathbf{y} \in \mathbb{B}^1(H)$. In particular, for all $\mathbf{y} \in \mathbb{B}^1(H)$, the indefinite integral

$$[0, T] \rightarrow H, \xi \mapsto \int_t^{t \vee \xi} \hat{b}(s, \mathbf{y}) ds$$

is continuous.

These considerations entails the well-posedness, for any fixed $\omega \in \Omega$, of the map

$$\psi: [0, T] \times \mathbb{B}^1(H) \times \mathbb{B}^1(H) \rightarrow \mathbb{B}^1(H) \quad (5.7)$$

defined by

$$\psi(t, \mathbf{x}, \mathbf{y}) := \mathbf{x}_{t \wedge \cdot} + \int_t^{t \vee \cdot} \hat{b}(s, \mathbf{y}) ds + (W_{t \vee \cdot}^B(\omega) - W_t^B(\omega)) \quad \forall (t, \mathbf{x}, \mathbf{y}) \in [0, T] \times \mathbb{B}^1(H) \times \mathbb{B}^1(H),$$

where W^B is a short notation for a fixed representant of $\int_0^\cdot B dW_s$.

In the following propositions, we prove existence and uniqueness of a fixed point for $\psi(t, \mathbf{x}, \cdot)$ and study how the fixed point depends on t, \mathbf{x} . We use standard arguments based on contractions in Banach spaces. What is important is that the SDE is here considered pathwise, in order to have better insight about the regularity of the paths $X^{t, \mathbf{x}}(\omega)$ with respect to \mathbf{x} .

Remark 5.4. In the notation ψ , the dependence on ω is not explicit. Nevertheless, we stress the very important fact that all the bounds for the Lipschitz constants and the differentials, which appear in the following propositions, are independent of ω . More precisely, the terms λ, α appearing in Proposition 5.5(i), the bounds for (5.9) and (5.10), the bounds for $\partial_{\mathbb{B}^1} \Lambda^{t, \cdot}$ and $\partial_{\mathbb{B}^1}^2 \Lambda^{t, \cdot}$ in Proposition 5.6, can be — and we assume that they are — chosen independently of ω .

For $\lambda > 0$, we introduce on $\mathbb{B}^1(H)$ the norm

$$|\mathbf{x}|_\lambda := \sup_{t \in [0, T]} e^{-\lambda t} |\mathbf{x}(t)|_H, \quad \forall \mathbf{x} \in \mathbb{B}^1(H).$$

Then $|\cdot|_\lambda$ is equivalent to $|\cdot|_\infty$.

Hereafter, we denote by $\mathbb{B}_\infty^1(H)$ the Banach space $(\mathbb{B}^1(H), |\cdot|_\infty)$ and by $\mathbb{B}_\lambda^1(H)$ the equivalent Banach space $(\mathbb{B}^1(H), |\cdot|_\lambda)$.

Proposition 5.5.

(i) *There exists $\lambda > 0$ and $\alpha \in (0, 1)$ such that*

$$\sup_{(t, \mathbf{x}) \in [0, T] \times \mathbb{B}^1(H)} |\psi(t, \mathbf{x}, \mathbf{y}) - \psi(t, \mathbf{x}, \mathbf{y}')|_\lambda \leq \alpha |\mathbf{y} - \mathbf{y}'|_\lambda \quad \forall \mathbf{y}, \mathbf{y}' \in \mathbb{B}^1(H). \quad (5.8)$$

(ii) *The restriction of ψ to $[0, T] \times \mathbb{W} \times \mathbb{B}_\infty^1(H)$ is \mathbb{W} -valued and continuous.*

(iii) *For all $t \in [0, T]$, the section*

$$\psi(t, \cdot, \cdot): \mathbb{B}_\infty^1(H) \times \mathbb{B}_\infty^1(H) \rightarrow \mathbb{B}_\infty^1(H), (\mathbf{x}, \mathbf{y}) \mapsto \psi(t, \mathbf{x}, \mathbf{y})$$

is strongly continuously Gâteaux differentiable up to order 2, i.e.

$$\psi(t, \cdot, \cdot) \in \mathcal{G}^2(\mathbb{B}_\infty^1(H) \times \mathbb{B}_\infty^1(H), \mathbb{B}_\infty^1(H)).$$

Moreover,

$$\sup_{\substack{t \in [0, T], \mathbf{x}, \mathbf{y} \in \mathbb{B}_\infty^1(H) \\ \mathbf{v} \in \mathbb{B}_\infty^1(H), |\mathbf{v}|_\infty \leq 1}} |\partial_2 \psi(t, \mathbf{x}, \mathbf{y}) \cdot \mathbf{v}|_\infty + \sup_{\substack{t \in [0, T], \mathbf{x}, \mathbf{y} \in \mathbb{B}_\infty^1(H) \\ \mathbf{v} \in \mathbb{B}_\infty^1(H), |\mathbf{v}|_\infty \leq 1}} |\partial_3 \psi(t, \mathbf{x}, \mathbf{y}) \cdot \mathbf{v}|_\infty < \infty \quad (5.9)$$

$$\sup_{\substack{t \in [0, T], \mathbf{x}, \mathbf{y} \in \mathbb{B}_\infty^1(H) \\ \mathbf{v}, \mathbf{w} \in \mathbb{B}_\infty^1(H), |\mathbf{v}|_\infty \vee |\mathbf{w}|_\infty \leq 1}} |\partial_3^2 \psi(t, \mathbf{x}, \mathbf{y}) \cdot (\mathbf{v}, \mathbf{w})|_\infty < \infty, \quad (5.10)$$

where $\partial_i \psi$ and $\partial_i^2 \psi$ denote the first- and second-order Gâteaux differential of ψ with respect to the i -th variable.

(iv) If $t_n \rightarrow t$ in $[0, T]$, $\mathbf{x}_n \rightarrow \mathbf{x}$ in $\mathbb{B}_\infty^1(H)$, $\mathbf{y}_n \rightarrow \mathbf{y}$ in $\mathbb{B}_{\sigma^s}^1(H)$, $\mathbf{v}_n \rightarrow \mathbf{v}$ in $\mathbb{B}_{\sigma^s}^1(H)$, $\mathbf{w}_n \rightarrow \mathbf{w}$ in $\mathbb{B}_{\sigma^s}^1(H)$, then

$$\partial_3 \psi(t_n, \mathbf{x}_n, \mathbf{y}_n) \cdot \mathbf{v}_n \rightarrow \partial_3 \psi(t, \mathbf{x}, \mathbf{y}) \cdot \mathbf{v} \text{ in } \mathbb{B}_\infty^1(H) \quad (5.11)$$

$$\partial_3^2 \psi(t_n, \mathbf{x}_n, \mathbf{y}_n) \cdot (\mathbf{v}_n, \mathbf{w}_n) \rightarrow \partial_3^2 \psi(t, \mathbf{x}, \mathbf{y}) \cdot (\mathbf{v}, \mathbf{w}) \text{ in } \mathbb{B}_\infty^1(H). \quad (5.12)$$

Proof. (i) For $t \in [0, T]$ and $\mathbf{x} \in \mathbb{B}^1(H)$, by standard computations, we have

$$\begin{aligned} e^{-\lambda s} |\psi(t, \mathbf{x}, \mathbf{y})(s) - \psi(t, \mathbf{x}, \mathbf{y}')(s)|_H &\leq e^{-\lambda s} \int_0^s |\hat{b}(r, \mathbf{y}) - \hat{b}(r, \mathbf{y}')|_H dr \\ &\leq \int_0^s e^{-\lambda(s-r)} e^{-\lambda r} |\hat{b}(r, \mathbf{y}) - \hat{b}(r, \mathbf{y}')|_H dr \\ &\leq \frac{1 - e^{-\lambda T}}{\lambda} |\hat{b}(\cdot, \mathbf{y}) - \hat{b}(\cdot, \mathbf{y}')|_\lambda \\ &\leq \frac{1 - e^{-\lambda T}}{\lambda} N |\mu|_1 |\mathbf{y} - \mathbf{y}'|_\lambda, \end{aligned}$$

for all $\mathbf{y}, \mathbf{y}' \in \mathbb{B}^1(H)$ and all $s \in [0, T]$. Then, for all $t, \mathbf{x}, \mathbf{y}, \mathbf{y}'$,

$$|\psi(t, \mathbf{x}, \mathbf{y}) - \psi(t, \mathbf{x}, \mathbf{y}')|_\lambda \leq \frac{1 - e^{-\lambda T}}{\lambda} N |\mu|_1 |\mathbf{y} - \mathbf{y}'|_\lambda. \quad (5.13)$$

By defining $\alpha := \frac{1 - e^{-\lambda T}}{\lambda} N |\mu|_1$, for λ sufficiently large we obtain (i).

(ii) Due to (i), it is sufficient to prove that $\psi(\cdot, \cdot, \mathbf{y})$ is \mathbb{W} -valued and continuous on $[0, T] \times \mathbb{W}$, for all $\mathbf{y} \in \mathbb{B}^1(H)$. But this comes from the continuity of the maps

$$[0, T] \times \mathbb{W} \rightarrow \mathbb{W}, (t, \mathbf{x}) \mapsto \mathbf{x}_{t \wedge \cdot}, \quad [0, T] \rightarrow H, s \mapsto \int_0^s \hat{b}(r, \mathbf{y}) dr + W_s^B(\omega).$$

(iii)+(iv) We begin by showing that, for all $t \in [0, T]$,

$$\Psi_t : \mathbb{B}_\infty^1(H) \rightarrow \mathbb{B}_\infty^1(H), \mathbf{y} \mapsto \int_t^{t \vee \cdot} \hat{b}(r, \mathbf{y}) dr \quad (5.14)$$

is strongly continuously Gâteaux differentiable up to order 2, with bounded differentials (bound uniform in t). By standard computations, due to Assumption 5.1(iii), we have

$$\begin{aligned} h^{-1}(\Psi_t(\mathbf{y} + h\mathbf{v}) - \Psi_t(\mathbf{y})) \\ = \int_t^{t \vee \cdot} \left(\int_0^1 \langle \nabla_H b \left(r, \int_{[0, T]} \tilde{\mathbf{y}}(r-s) \mu(ds) + \theta h \int_{[0, T]} \tilde{\mathbf{v}}(r-s) \mu(ds) \right), \int_{[0, T]} \tilde{\mathbf{v}}(r-s) \mu(ds) \rangle_H d\theta \right) dr, \end{aligned}$$

where $\nabla_H b$ represents $\partial_H b$ in H . Due to the assumptions on $\partial_H b$, we can pass to the limit $h^{-1}(\Psi_t(\mathbf{y} + h\mathbf{v}) - \Psi_t(\mathbf{y}))$ in $\mathbb{B}_\infty^1(H)$ as $h \rightarrow 0$ and obtain

$$\partial \Psi_t(\mathbf{y}) \cdot \mathbf{v} = \int_t^{t \vee \cdot} \langle \nabla_H b \left(r, \int_{[0, T]} \tilde{\mathbf{y}}(r-s) \mu(ds) \right), \int_{[0, T]} \tilde{\mathbf{v}}(r-s) \mu(ds) \rangle_H dr. \quad (5.15)$$

Notice that, if $\mathbf{y}_n \rightarrow \mathbf{y}$ and $\mathbf{v}_n \rightarrow \mathbf{v}$ in $\mathbb{B}_{\sigma^s}^1(H)$, then, by Proposition 2.1(i),

$$\int_{[0, T]} \tilde{\mathbf{y}}_n(r-s) \mu(ds) \rightarrow \int_{[0, T]} \tilde{\mathbf{y}}(r-u) \mu(du) \quad \forall r \in [0, T]$$

and the family

$$\left\{ \int_{[0,T]} \tilde{\mathbf{y}}_n(r-u) \mu(du) \right\}_{\substack{r \in [0,T] \\ n \in \mathbb{N}}}$$

is bounded in H . The same holds with respect to $\tilde{\mathbf{v}}_n$ and $\tilde{\mathbf{v}}$. If $t_n \rightarrow t$ in $[0, T]$, by strong continuity of $\partial_H b$ and using (5.15), we conclude that

$$|\partial \Psi_t(\mathbf{y}).\mathbf{v} - \partial \Psi_{t_n}(\mathbf{y}_n).\mathbf{v}_n|_\infty \rightarrow 0. \quad (5.16)$$

This proves (5.11), because $\partial_3 \psi(t, \mathbf{x}, \mathbf{y}) = \partial \Psi_t(\mathbf{y})$ for all $(t, \mathbf{x}, \mathbf{y}) \in [0, T] \times \mathbb{B}^1(H) \times \mathbb{B}^1(H)$. In particular, the limit (5.16) holds when $t_n = t$, for all $n \in \mathbb{N}$, and the convergences $\mathbf{y}_n \rightarrow \mathbf{y}$ and $\mathbf{v}_n \rightarrow \mathbf{v}$ take place in $\mathbb{B}_\infty^1(H)$. This shows that $\Psi_t \in \mathcal{G}^1(\mathbb{B}_\infty^1(H), \mathbb{B}_\infty^1(H))$, and, by (5.3) and (5.15), that the first order differentials are bounded, with bound uniform in t . By observing that $\partial_2 \psi(t, \mathbf{x}, \mathbf{y}).\mathbf{v} = \mathbf{v}_{t \wedge \cdot}$ for all $t \in [0, T]$, $\mathbf{x}, \mathbf{y}, \mathbf{v} \in \mathbb{B}^1(H)$, we have then proved that $\psi(t, \cdot, \cdot) \in \mathcal{G}^1(\mathbb{B}_\infty^1(H) \times \mathbb{B}_\infty^1(H), \mathbb{B}_\infty^1(H))$ and that (5.9) holds true.

Regarding the second order derivative, by using similar arguments as above, we obtain

$$\begin{aligned} \partial^2 \Psi_t(\mathbf{x}).(\mathbf{v}, \mathbf{w}) &= \\ &= \int_t^{t \vee \cdot} \partial_H^2 b \left(r, \int_{[0,T]} \tilde{\mathbf{y}}(r-s) \mu(ds) \right) \cdot \left(\int_{[0,T]} \tilde{\mathbf{v}}(r-s) \mu(ds), \int_{[0,T]} \tilde{\mathbf{w}}(r-s) \mu(ds) \right) dr \end{aligned} \quad (5.17)$$

and the continuity of

$$[0, T] \times \mathbb{B}_{\sigma^s}^1(H) \times \mathbb{B}_{\sigma^s}^1(H) \times \mathbb{B}_{\sigma^s}^1(H) \rightarrow \mathbb{B}_\infty^1(H), (t, \mathbf{y}, \mathbf{v}, \mathbf{w}) \mapsto \partial^2 \Psi_t(\mathbf{y}).(\mathbf{v}, \mathbf{w}).$$

Then, since $\partial_2^2 \psi(t, \mathbf{x}, \mathbf{y}) = 0$ and $\partial_3^2 \psi(t, \mathbf{x}, \mathbf{y}) = \partial^2 \Psi_t(\mathbf{y})$, $\psi(t, \cdot, \cdot) \in \mathcal{G}^2(\mathbb{B}_\infty^1(H) \times \mathbb{B}_\infty^1(H), \mathbb{B}_\infty^1(H))$. By (5.4), also the second order differentials $\partial_2^2 \psi, \partial_3^2 \psi$ are bounded, with bound uniform in t . \blacksquare

In the following proposition we see how the regularity properties of ψ are inherited by the associated fixed-point map.

Proposition 5.6.

(i) For all $(t, \mathbf{x}) \in [0, T] \times \mathbb{B}^1(H)$, there exists a unique $\Lambda^{t, \mathbf{x}} \in \mathbb{B}^1(H)$ such that

$$\Lambda^{t, \mathbf{x}} = \psi(t, \mathbf{x}, \Lambda^{t, \mathbf{x}}).$$

(ii) The map

$$\Lambda: [0, T] \times \mathbb{B}_\infty^1(H) \rightarrow \mathbb{B}_\infty^1(H), (t, \mathbf{x}) \mapsto \Lambda^{t, \mathbf{x}}$$

is Lipschitz in \mathbf{x} , with a bound for the Lipschitz constant independent of t .

(iii) The restriction of Λ to $[0, T] \times \mathbb{W}$ is continuous and \mathbb{W} -valued.

(iv) For all $t \in [0, T]$, $\Lambda^{t, \cdot} \in \mathcal{G}^2(\mathbb{B}_\infty^1(H), \mathbb{B}_\infty^1(H))$ and $\partial_{\mathbb{B}^1} \Lambda^{t, \cdot}, \partial_{\mathbb{B}^1}^2 \Lambda^{t, \cdot}$ are uniformly bounded, uniformly in t .

(v) For all $t \in [0, T]$ and $\mathbf{x} \in \mathbb{B}^1(H)$, $I - \partial_3 \psi(t, \mathbf{x}, \Lambda^{t, \mathbf{x}}) \in L(\mathbb{B}_\infty^1(H))$ is invertible and

$$\mathbb{B}_\infty^1(H) \rightarrow L(\mathbb{B}_\infty^1(H)), \mathbf{x} \mapsto (I - \partial_3 \psi(t, \mathbf{x}, \Lambda^{t, \mathbf{x}}))^{-1} \quad (5.18)$$

is strongly continuous.

(vi) For all $t \in [0, T]$, $\mathbf{x}, \mathbf{v}, \mathbf{w} \in \mathbb{B}^1(H)$, we have

$$\begin{aligned} \partial_{\mathbb{B}^1} \Lambda^{t, \mathbf{x}} \cdot \mathbf{v} &= (I - \partial_3 \psi(t, \mathbf{x}, \Lambda^{t, \mathbf{x}}))^{-1} (\partial_2 \psi(t, \mathbf{x}, \Lambda^{t, \mathbf{x}}) \cdot \mathbf{v}) \\ \partial_{\mathbb{B}^1}^2 \Lambda^{t, \mathbf{x}} \cdot (\mathbf{v}, \mathbf{w}) &= (I - \partial_3 \psi(t, \mathbf{x}, \Lambda^{t, \mathbf{x}}))^{-1} (\partial_3^2 \psi(t, \mathbf{x}, \Lambda^{t, \mathbf{x}}) \cdot ((\partial_{\mathbb{B}^1} \Lambda^{t, \mathbf{x}} \cdot \mathbf{v}), (\partial_{\mathbb{B}^1} \Lambda^{t, \mathbf{x}} \cdot \mathbf{w}))) \end{aligned}$$

Proof. By Proposition 5.5(i), we can choose $\lambda > 0$ such that $\psi(t, \mathbf{x}, \cdot)$ is an α -contraction on $\mathbb{B}_\lambda^1(H)$, with $\alpha \in (0, 1)$, uniformly in $(t, \mathbf{x}) \in [0, T] \times \mathbb{B}^1(H)$.

(i) Apply Banach's contraction principle to $\psi(t, \mathbf{x}, \cdot)$ on $\mathbb{B}_\lambda^1(H)$.

(ii) For every $t \in [0, T]$, we have

$$|\psi(t, \mathbf{x}, \mathbf{y}) - \psi(t, \mathbf{x}', \mathbf{y})|_\lambda \leq |\mathbf{x} - \mathbf{x}'|_\lambda \quad \forall \mathbf{x}, \mathbf{x}' \in \mathbb{B}^1(H).$$

The conclusion follows by [20, p. 13, inequality (* * *)].

(iii) Since ψ maps $[0, T] \times \mathbb{W} \times \mathbb{W}$ into \mathbb{W} by Proposition 5.5(ii), we also have that Λ maps $[0, T] \times \mathbb{W}$ into \mathbb{W} . Let us denote by $\Lambda_{\mathbb{W}}$ the map

$$\Lambda_{\mathbb{W}}: [0, T] \times \mathbb{W} \rightarrow \mathbb{W}, (t, \mathbf{x}) \mapsto \Lambda^{t, \mathbf{x}}.$$

By (ii), to prove the continuity of $\Lambda_{\mathbb{W}}$, it is sufficient to show the continuity of $\Lambda^{t, \mathbf{x}}$, for fixed $\mathbf{x} \in \mathbb{W}$. Let $t_n \rightarrow t$ in $[0, T]$. We have

$$\psi(t_n, \mathbf{x}, \mathbf{y}) \rightarrow \psi(t, \mathbf{x}, \mathbf{y}) \text{ in } \mathbb{W}, \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{W}.$$

Then the conclusion follows by [7, Theorem 7.1.5].

(iv)+(v)+(vi) Thanks to Proposition 5.5(iii), we can apply [7, Theorems 7.1.2 and 7.1.3] to all maps $\psi(t, \cdot, \cdot)$, for all $t \in [0, T]$. This shows (iv) and (vi).

It remains only to comment the strong continuity of (5.18) (which is indeed contained in the proof of [7, Theorems 7.1.2 and 7.1.3]). This comes from the fact that, for all $t \in [0, T]$, by (ii) and Proposition 5.5(iii), the map

$$\mathbb{B}_\lambda^1(H) \rightarrow L(\mathbb{B}_\lambda^1(H)), \mathbf{x} \mapsto \partial_3 \psi(t, \mathbf{x}, \Lambda^{t, \mathbf{x}})$$

is strongly continuous and $|\partial_3 \psi(t, \mathbf{x}, \Lambda^{t, \mathbf{x}})|_{L(\mathbb{B}_\lambda^1(H))} \leq \alpha$ for all $\mathbf{x} \in \mathbb{B}^1(H)$. By writing

$$(I - \partial_3 \psi(t, \mathbf{x}, \Lambda^{t, \mathbf{x}}))^{-1} \mathbf{v} = \sum_{n \in \mathbb{N}} (\partial_3 \psi(t, \mathbf{x}, \Lambda^{t, \mathbf{x}}))^n \mathbf{v} \quad \forall \mathbf{v} \in \mathbb{B}^1(H) \quad (5.19)$$

and by Lebesgue's dominated convergence theorem (for sums), we see the strong continuity of

$$\mathbb{B}_\lambda^1(H) \rightarrow L(\mathbb{B}_\lambda^1(H)), \mathbf{x} \mapsto (I - \partial_3 \psi(t, \mathbf{x}, \Lambda^{t, \mathbf{x}}))^{-1}. \quad \blacksquare$$

The following proposition provides the good continuity of the differentials of Λ with respect to \mathbf{x} , that we will later need in order to apply Theorem 4.2 and Corollary 4.3 when the process X has the dynamics (5.6).

Proposition 5.7. *Let $t \in [0, T]$.*

(i) *If $\mathbf{x}_n \rightarrow \mathbf{x}$ in $\mathbb{B}_\infty^1(H)$, $\mathbf{v}_n \rightarrow \mathbf{v}$ in $\mathbb{B}_{\sigma^s}^1(H)$, and $\mathbf{w}_n \rightarrow \mathbf{w}$ in $\mathbb{B}_{\sigma^s}^1(H)$, then*

$$\partial_{\mathbb{B}^1} \Lambda^{t, \mathbf{x}_n} \cdot \mathbf{v}_n \rightarrow \partial_{\mathbb{B}^1} \Lambda^{t, \mathbf{x}} \cdot \mathbf{v} \text{ in } \mathbb{B}_{\sigma^s}^1(H) \quad (5.20)$$

$$\partial_{\mathbb{B}^1}^2 \Lambda^{t, \mathbf{x}_n} \cdot (\mathbf{v}_n, \mathbf{w}_n) \rightarrow \partial_{\mathbb{B}^1}^2 \Lambda^{t, \mathbf{x}} \cdot (\mathbf{v}, \mathbf{w}) \text{ in } \mathbb{B}_\infty^1(H). \quad (5.21)$$

(ii) *If $t_n \rightarrow t^+$ in $[0, T]$, $\mathbf{x} \in \mathbb{W}$, $\mathbf{v}, \mathbf{w} \in \mathbb{B}^1(H)$, then*

$$\partial_{\mathbb{B}^1} \Lambda^{t_n, \mathbf{x}} \cdot \mathbf{v} \rightarrow \partial_{\mathbb{B}^1} \Lambda^{t, \mathbf{x}} \cdot \mathbf{v} \text{ in } \mathbb{B}_{\sigma^s}^1(H) \quad (5.22)$$

$$\partial_{\mathbb{B}^1}^2 \Lambda^{t_n, \mathbf{x}} \cdot (\mathbf{v}, \mathbf{w}) \rightarrow \partial_{\mathbb{B}^1}^2 \Lambda^{t, \mathbf{x}} \cdot (\mathbf{v}, \mathbf{w}) \text{ in } \mathbb{W}. \quad (5.23)$$

Proof. (i) Let $t \in [0, T]$, $\mathbf{x}_n \rightarrow \mathbf{x}$ in $\mathbb{B}_\infty^1(H)$, $\mathbf{v}_n \rightarrow \mathbf{v}$ in $\mathbb{B}_{\sigma^s}^1(H)$. By Proposition 2.1(i), $\{\mathbf{v}_n\}_{n \in \mathbb{N}}$ is bounded in $\mathbb{B}_\infty^1(H)$. By Proposition 5.6(iv), $\{\partial_{\mathbb{B}^1} \Lambda^{t, \mathbf{x}_n} \cdot \mathbf{v}_n\}_{n \in \mathbb{N}, t \in [0, T]}$ is bounded in $\mathbb{B}_\infty^1(H)$. In particular, $\{\partial_{\mathbb{B}^1} \Lambda^{t, \mathbf{x}_n} \cdot \mathbf{v}_n\}_{n \in \mathbb{N}}$ is bounded in the Hilbert space $L^2([0, T], H)$, which is separable. Then we can find subsequences $\{\mathbf{x}_{n_k}\}_{k \in \mathbb{N}}$ and $\{\mathbf{v}_{n_k}\}_{k \in \mathbb{N}}$ such that

$$\lim_{k \rightarrow \infty} \partial_{\mathbb{B}^1} \Lambda^{t, \mathbf{x}_{n_k}} \cdot \mathbf{v}_{n_k} = Z \text{ weakly in } L^2([0, T], H),$$

for some $Z \in L^2([0, T], H)$. We recall that $\partial_3 \psi(t, \mathbf{x}', \mathbf{y}') = \partial \Psi_t(\mathbf{y}')$ for all $\mathbf{x}', \mathbf{y}' \in \mathbb{B}^1(H)$, where Ψ_t was defined in the proof of Proposition 5.5 by (5.14). By (5.15), for $\mathbf{y}', \mathbf{v}' \in \mathbb{B}^1(H)$, we have

$$\begin{aligned} (\partial \Psi_t(\mathbf{y}')) \cdot \mathbf{v}'(\xi) &= \int_t^{t \vee \xi} \left\langle \nabla_y b \left(r, \int_{[0, T]} \tilde{\mathbf{y}}'(r-u) \mu(du) \right), \int_{[0, T]} \tilde{\mathbf{v}}'(r-u) \mu(du) \right\rangle_H dr \\ &= \int_t^{t \vee \xi} \left(\int_{[0, T]} \left\langle \nabla_y b \left(r, \int_{[0, T]} \tilde{\mathbf{y}}'(r-u) \mu(du) \right), \tilde{\mathbf{v}}'(r-s) \right\rangle_H \mu(ds) \right) dr \\ &= \int_{[0, T]} \left(\int_t^{t \vee \xi} \left\langle \nabla_y b \left(r, \int_{[0, T]} \tilde{\mathbf{y}}'(r-u) \mu(du) \right), \tilde{\mathbf{v}}'(r-s) \right\rangle_H dr \right) \mu(ds). \end{aligned}$$

By replacing \mathbf{x}' by \mathbf{x}_{n_k} , \mathbf{y}' by $\Lambda^{t, \mathbf{x}_{n_k}}$, and \mathbf{v}' by $\partial_{\mathbb{B}^1} \Lambda^{t, \mathbf{x}_{n_k}} \cdot \mathbf{v}_{n_k}$, we obtain ⁽²⁾

$$\begin{aligned} &(\partial_3 \psi(t, \mathbf{x}_{n_k}, \Lambda^{t, \mathbf{x}_{n_k}}) \cdot (\partial_{\mathbb{B}^1} \Lambda^{t, \mathbf{x}_{n_k}} \cdot \mathbf{v}_{n_k}))(\xi) \\ &= \int_{[0, T]} \left(\int_t^{t \vee \xi} \left\langle \nabla_y b \left(r, \int_{[0, T]} (\Lambda^{t, \mathbf{x}_{n_k}})^\sim(r-u) \mu(du) \right), (\partial_{\mathbb{B}^1} \Lambda^{t, \mathbf{x}_{n_k}} \cdot \mathbf{v}_{n_k})^\sim(r-s) \right\rangle_H dr \right) \mu(ds). \end{aligned} \quad (5.24)$$

Due to the fact that $\{\partial_{\mathbb{B}^1} \Lambda^{t, \mathbf{x}_{n_k}} \cdot \mathbf{v}_{n_k}\}_{k \in \mathbb{N}}$ is uniformly bounded in $\mathbb{B}_\infty^1(H)$, passing to another subsequence if necessary, we can assume that $(\partial_{\mathbb{B}^1} \Lambda^{t, \mathbf{x}_{n_k}} \cdot \mathbf{v}_{n_k})(0)$ is weakly convergent in H to some $z_0 \in H$. Then

$$\lim_{k \rightarrow \infty} (\partial_{\mathbb{B}^1} \Lambda^{t, \mathbf{x}_{n_k}} \cdot \mathbf{v}_{n_k})^\sim = \mathbf{1}_{[-T, 0)}(\cdot) z_0 + \mathbf{1}_{[0, T]} Z \text{ weakly in } L^2([-T, T], H). \quad (5.25)$$

²If the argument \mathbf{y} of the notation $\tilde{\mathbf{y}}$ is long, we write $(\mathbf{y})^\sim$.

By Proposition 5.6(ii), we have

$$\Lambda^{t, \mathbf{x}_{n_k}} \rightarrow \Lambda^{t, \mathbf{x}} \text{ in } \mathbb{B}_\infty^1(H),$$

then, since $b(r, \cdot) \in C_b^1(H, H)$ (see Remark 5.2),

$$\lim_{k \rightarrow \infty} \left| \nabla_y b \left(r, \int_{[0, T]} (\Lambda^{t, \mathbf{x}_{n_k}})^{\sim}(r-u) \mu(du) \right) - \nabla_y b \left(r, \int_{[0, T]} (\Lambda^{t, \mathbf{x}})^{\sim}(r-u) \mu(du) \right) \right|_H = 0,$$

for all $r \in [0, T]$. In particular, by Lebesgue's dominated convergence theorem,

$$\nabla_y b \left(\cdot, \int_{[0, T]} (\Lambda^{t, \mathbf{x}_{n_k}})^{\sim}(\cdot-u) \mu(du) \right) \rightarrow \nabla_y b \left(\cdot, \int_{[0, T]} (\Lambda^{t, \mathbf{x}})^{\sim}(\cdot-u) \mu(du) \right) \quad (5.26)$$

strongly in $L^2([0, T], H)$. By (5.25) and (5.26), we have, for all $s, \xi \in [0, T]$,

$$\begin{aligned} \lim_{k \rightarrow \infty} \int_t^{t \vee \xi} \langle \nabla_y b \left(r, \int_{[0, T]} (\Lambda^{t, \mathbf{x}_{n_k}})^{\sim}(r-u) \mu(du) \right), (\partial_{\mathbb{B}^1} \Lambda(t, \mathbf{x}_{n_k}) \cdot \mathbf{v}_{n_k})^{\sim}(r-s) \rangle_H dr = \\ = \int_t^{t \vee \xi} \langle \nabla_y b \left(r, \int_{[0, T]} (\Lambda^{t, \mathbf{x}})^{\sim}(r-u) \mu(du) \right), \mathbf{1}_{[-T, 0)}(r-s) z_0 + \mathbf{1}_{[0, T]}(r-s) Z \rangle_H dr. \end{aligned}$$

Since the latter limit holds for all ξ and s , by (5.24) we have that the limit

$$\lim_{k \rightarrow \infty} (\partial_3 \psi(t, \mathbf{x}_{n_k}, \Lambda^{t, \mathbf{x}_{n_k}}) \cdot (\partial_{\mathbb{B}^1} \Lambda^{t, \mathbf{x}_{n_k}} \cdot \mathbf{v}_{n_k}))(\xi)$$

exists for all $\xi \in [0, T]$. By Proposition 2.1(i), since the sequence is uniformly bounded, we can finally conclude that

$$\{\partial_3 \psi(t, \mathbf{x}_{n_k}, \Lambda^{t, \mathbf{x}_{n_k}}) \cdot (\partial_{\mathbb{B}^1} \Lambda^{t, \mathbf{x}_{n_k}} \cdot \mathbf{v}_{n_k})\}_{k \in \mathbb{N}}$$

converges in $\mathbb{B}_{\sigma^s}^1(H)$. Now we are almost done. By Proposition 5.6(vi), we have

$$\begin{aligned} \partial_{\mathbb{B}^1} \Lambda^{t, \mathbf{x}_{n_k}} \cdot \mathbf{v}_{n_k} &= \partial_3 \psi(t, \mathbf{x}_{n_k}, \Lambda^{t, \mathbf{x}_{n_k}}) \cdot (\partial_{\mathbb{B}^1} \Lambda^{t, \mathbf{x}_{n_k}} \cdot \mathbf{v}_{n_k}) + \partial_2 \psi(t, \mathbf{x}_{n_k}, \Lambda^{t, \mathbf{x}_{n_k}}) \cdot \mathbf{v}_{n_k} \\ &= \partial_3 \psi(t, \mathbf{x}_{n_k}, \Lambda^{t, \mathbf{x}_{n_k}}) \cdot (\partial_{\mathbb{B}^1} \Lambda^{t, \mathbf{x}_{n_k}} \cdot \mathbf{v}_{n_k}) + (\mathbf{v}_{n_k})_{t \wedge \cdot}. \end{aligned} \quad (5.27)$$

By considering what proved above and the assumptions on $\{\mathbf{v}_n\}_{n \in \mathbb{N}}$, there exists $\gamma \in \mathbb{B}^1(H)$ such that

$$\partial_3 \psi(t, \mathbf{x}_{n_k}, \Lambda^{t, \mathbf{x}_{n_k}}) \cdot (\partial_{\mathbb{B}^1} \Lambda^{t, \mathbf{x}_{n_k}} \cdot \mathbf{v}_{n_k}) + (\mathbf{v}_{n_k})_{t \wedge \cdot} \rightarrow \gamma \text{ in } \mathbb{B}_{\sigma^s}^1(H). \quad (5.28)$$

Then, (5.27), (5.28), and Proposition 5.5(iv), we have

$$\partial_3 \psi(t, \mathbf{x}_{n_k}, \Lambda^{t, \mathbf{x}_{n_k}}) \cdot (\partial_{\mathbb{B}^1} \Lambda^{t, \mathbf{x}_{n_k}} \cdot \mathbf{v}_{n_k}) \rightarrow \partial_3 \psi(t, \mathbf{x}, \Lambda^{t, \mathbf{x}}) \cdot \gamma \text{ in } \mathbb{B}_\infty^1(H), \text{ hence in } \mathbb{B}_{\sigma^s}^1(H).$$

By taking the limit in $\mathbb{B}_{\sigma^s}^1(H)$ in (5.27), we have

$$\gamma = \partial_3 \psi(t, \mathbf{x}, \Lambda^{t, \mathbf{x}}) \cdot \gamma + \mathbf{v}_{t \wedge \cdot},$$

which entails $\gamma = \partial_{\mathbb{B}^1} \Lambda^{t, \mathbf{x}} \cdot \mathbf{v}$, by Proposition 5.6(vi). This shows that

$$\partial_{\mathbb{B}^1} \Lambda^{t, \mathbf{x}_{n_k}} \cdot \mathbf{v}_{n_k} \rightarrow \partial_{\mathbb{B}^1} \Lambda^{t, \mathbf{x}} \cdot \mathbf{v} \text{ in } \mathbb{B}_{\sigma^s}^1(H).$$

Since the original sequences $\{\mathbf{x}\}_{n \in \mathbb{N}}, \{\mathbf{v}_n\}_{n \in \mathbb{N}}$ were arbitrary, (5.20) is proved.

To prove (5.21), we use Proposition 5.6(vi). But now most of the work is done. By (5.20), we have

$$\begin{aligned}\partial_{\mathbb{B}^1} \Lambda^{t, \mathbf{x}_n} \cdot \mathbf{v}_n &\rightarrow \partial_{\mathbb{B}^1} \Lambda^{t, \mathbf{x}} \cdot \mathbf{v} \text{ in } \mathbb{B}_{\sigma^s}^1(H) \\ \partial_{\mathbb{B}^1} \Lambda^{t, \mathbf{x}_n} \cdot \mathbf{w}_n &\rightarrow \partial_{\mathbb{B}^1} \Lambda^{t, \mathbf{x}} \cdot \mathbf{w} \text{ in } \mathbb{B}_{\sigma^s}^1(H).\end{aligned}$$

By Proposition 5.5(iv), we have

$$\begin{aligned}\lim_{k \rightarrow \infty} \partial_3^2 \psi(t, \mathbf{x}_n, \Lambda^{t, \mathbf{x}_n}) \cdot ((\partial_{\mathbb{B}^1} \Lambda^{t, \mathbf{x}_n} \cdot \mathbf{v}_n), (\partial_{\mathbb{B}^1} \Lambda^{t, \mathbf{x}_n} \cdot \mathbf{w}_n)) &= \\ &= \partial_3^2 \psi(t, \mathbf{x}, \Lambda^{t, \mathbf{x}}) \cdot ((\partial_{\mathbb{B}^1} \Lambda^{t, \mathbf{x}} \cdot \mathbf{v}), (\partial_{\mathbb{B}^1} \Lambda^{t, \mathbf{x}} \cdot \mathbf{w}))\end{aligned}$$

where the limit is taken in $\mathbb{B}_{\infty}^1(H)$. We can now conclude by using the strong continuity claimed in Proposition 5.6(v) and the formula for the second order derivative provided by Proposition 5.6(vi).

(ii) Let $t_n \rightarrow t^+$ in $[0, T]$, $\mathbf{x} \in \mathbb{W}$, $\mathbf{v} \in \mathbb{B}^1(H)$. By Proposition 5.6(vi) and by taking into account formula (5.19), we can write

$$\begin{aligned}\partial_{\mathbb{B}^1} \Lambda^{t_n, \mathbf{x}} \cdot \mathbf{v} &= (I - \partial_3 \psi(t, \mathbf{x}, \Lambda^{t, \mathbf{x}}))^{-1} (\partial_2 \psi(t_n, \mathbf{x}, \Lambda^{t_n, \mathbf{x}}) \cdot \mathbf{v}) \\ &= \sum_{k \in \mathbb{N}} (\partial_3 \psi(t_n, \mathbf{x}, \Lambda^{t_n, \mathbf{x}}))^k \mathbf{v}_{t_n \wedge \cdot} = \mathbf{v}_{t_n \wedge \cdot} + \sum_{k \geq 1} (\partial_3 \psi(t_n, \mathbf{x}, \Lambda^{t_n, \mathbf{x}}))^k \mathbf{v}_{t_n \wedge \cdot}.\end{aligned}$$

The fact that $t_n \rightarrow t$ from the right assures that $\mathbf{v}_{t_n \wedge \cdot} \rightarrow \mathbf{v}_{t \wedge \cdot}$ in $\mathbb{B}_{\sigma^s}^1(H)$. Moreover, by Proposition 5.6(iii), $\Lambda^{t_n, \mathbf{x}} \rightarrow \Lambda^{t, \mathbf{x}}$ in \mathbb{W} . Then, by Proposition 5.5(iii), (iv), and Lebesgue's dominated convergence theorem for sums, we have

$$\sum_{k \geq 1} (\partial_3 \psi(t_n, \mathbf{x}, \Lambda^{t_n, \mathbf{x}}))^k \mathbf{v}_{t_n \wedge \cdot} \rightarrow \sum_{k \geq 1} (\partial_3 \psi(t, \mathbf{x}, \Lambda^{t, \mathbf{x}}))^k \mathbf{v}_{t \wedge \cdot} \text{ in } \mathbb{B}_{\infty}^1(H).$$

Then $\partial_{\mathbb{B}^1} \Lambda^{t_n, \mathbf{x}} \cdot \mathbf{v} \rightarrow \partial_{\mathbb{B}^1} \Lambda^{t, \mathbf{x}} \cdot \mathbf{v}$ in $\mathbb{B}_{\sigma^s}^1(H)$ and (5.22) is proved.

Regarding (5.23), the argument is similar, by using the expression for $\partial_{\mathbb{B}^1}^2 \Lambda^{t_n, \mathbf{x}} \cdot (\mathbf{v}, \mathbf{w})$ provided by Proposition 5.6(vi), the convergence (5.22) just proved, and (5.12) in Proposition 5.5(iv) \blacksquare

We defined ψ for a given, fixed, $\omega \in \Omega$ (p. 27). For every such ψ , Propositions 5.5, 5.6, 5.7 apply. We can then define the map

$$\Omega \times [0, T] \times \mathbb{B}^1(H) \rightarrow \mathbb{B}^1(H), (\omega, t, \mathbf{x}) \mapsto X^{t, \mathbf{x}}(\omega) \quad (5.29)$$

where $X^{t, \mathbf{x}}(\omega)$ is the function $\Lambda^{t, \mathbf{x}}$ provided by Proposition 5.6, when ψ is associated to ω . It should be clear that $X^{t, \mathbf{x}}$ is the unique strong solution to SDE (5.6) in $\mathcal{L}_{\mathcal{P}_T}^0(\mathbb{W})$.

Let $f: \mathbb{B}^1(H) \rightarrow \mathbb{R}$ be a function. Hereafter, we assume that f satisfies the following assumption.

Assumption 5.8.

- (i) $f \in \mathcal{G}^2(\mathbb{B}_{\infty}^1(H), \mathbb{R})$;

(ii) the differentials ∂f and $\partial^2 f$ are bounded;

(iii) $\mathbb{B}_\infty^1(H) \times \mathbb{B}_{\sigma^s}^1(H) \rightarrow \mathbb{R}$, $(\mathbf{x}, \mathbf{v}) \mapsto \partial f(\mathbf{x}).\mathbf{v}$ is sequentially continuous;

(iv) $\mathbb{B}_\infty^1(H) \times \mathbb{B}_{\sigma^s}^1(H) \times \mathbb{B}_{\sigma^s}^1(H) \rightarrow \mathbb{R}$, $(\mathbf{x}, \mathbf{v}, \mathbf{w}) \mapsto \partial^2 f(\mathbf{x}).(\mathbf{v}, \mathbf{w})$ is sequentially continuous.

The following theorem shows that the main results of Section 4 can be applied in the present framework.

Theorem 5.9. *Let X be the unique strong solution of (5.6) and let*

$$\varphi: [0, T] \times \mathbb{W} \rightarrow \mathbb{R}, (t, \mathbf{x}) \mapsto \mathbb{E}[f(X^{t, \mathbf{x}})].$$

Then φ verifies Assumption 3.3. Moreover, for all $t \in (0, T)$ and all $\mathbf{x} \in \mathbb{W}$,

$$\mathcal{D}_t^- \varphi(t, \mathbf{x}_{t \wedge \cdot}) + \overline{\partial_{\mathbb{W}} \varphi}(t, \mathbf{x}).(\mathbf{1}_{[t, T]} b(t, \mathbf{x})) + \frac{1}{2} \mathbf{T} \left[\overline{\partial_{\mathbb{W}}^2 \varphi}(t, \mathbf{x}), B \right] = 0 \quad (5.30)$$

and for all $t \in [0, T]$, $t' \in [t, T]$, $Y \in \mathcal{L}_{\mathcal{D}_T}^p(\mathbb{W})$, $p > 2$,

$$\varphi(t', X^{t, Y}) = \varphi(t, Y) + \int_t^{t'} \overline{\partial_{\mathbb{W}} \varphi}(s, X^{t, Y}).(\mathbf{1}_{[s, T]} b(s, X^{t, Y})) dW_s \quad \mathbb{P}\text{-a.e.} \quad (5.31)$$

Proof. It is sufficient to show that φ verifies Assumption 3.3(ii),(iii), since the remaining part of the theorem comes from Theorem 4.2 and Corollary 4.3.

We begin by verifying Assumption 3.3(ii). By Proposition 5.6(iv), for all $(\omega, t) \in \Omega \times [0, T]$, the map $\mathbf{x} \mapsto X^{t, \mathbf{x}}(\omega)$ belongs to $\mathcal{G}^2(\mathbb{B}_\infty^1(H), \mathbb{B}_\infty^1(H))$ and has differentials $\partial_{\mathbb{B}^1} X^{t, \cdot}(\omega)$ and $\partial_{\mathbb{B}^1}^2 X^{t, \cdot}(\omega)$ bounded, with bound uniform in ω, t (recall Remark 5.4). Then, since $f \in \mathcal{G}^2(\mathbb{B}_\infty^1(H), \mathbb{R})$ and ∂f and $\partial^2 f$ are uniformly bounded, the composition $\mathbf{x} \mapsto f(X^{t, \mathbf{x}}(\omega))$ belongs to $\mathcal{G}^2(\mathbb{B}_\infty^1(H), \mathbb{R})$ and has differentials $\partial_{\mathbb{B}^1} f(X^{t, \cdot}(\omega))$ and $\partial_{\mathbb{B}^1}^2 f(X^{t, \cdot}(\omega))$ bounded, with bound uniform in ω, t . We have

$$\partial_{\mathbb{B}^1} f(X^{t, \mathbf{x}}(\omega)).\mathbf{v} = \partial f(X^{t, \mathbf{x}}(\omega)).(\partial_{\mathbb{B}^1} X^{t, \mathbf{x}}(\omega).\mathbf{v}) \quad (5.32)$$

for all $t \in [0, T]$, $\omega \in \Omega$, $\mathbf{x}, \mathbf{v} \in \mathbb{B}^1(H)$, and

$$\begin{aligned} \partial_{\mathbb{B}^1}^2 f(X^{t, \mathbf{x}}(\omega)).(\mathbf{v}, \mathbf{w}) &= \\ &= \partial^2 f(X^{t, \mathbf{x}}(\omega)).((\partial_{\mathbb{B}^1} X^{t, \mathbf{x}}(\omega).\mathbf{v}).(\partial_{\mathbb{B}^1} X^{t, \mathbf{x}}(\omega).\mathbf{w})) + \partial f(X^{t, \mathbf{x}}(\omega)).(\partial_{\mathbb{B}^1}^2 X^{t, \mathbf{x}}(\omega).(\mathbf{v}, \mathbf{w})) \end{aligned} \quad (5.33)$$

for all $t \in [0, T]$, $\omega \in \Omega$, $\mathbf{x}, \mathbf{v}, \mathbf{w} \in \mathbb{B}^1(H)$. Since $\partial_{\mathbb{B}^1} f(X^{t, \mathbf{x}}(\omega))$ and $\partial_{\mathbb{B}^1}^2 f(X^{t, \mathbf{x}}(\omega))$ are bounded, with bound uniform in ω, t, \mathbf{x} , we can easily see that

$$\partial_{\mathbb{B}^1} \varphi(t, \mathbf{x}).\mathbf{v} = \mathbb{E}[\partial_{\mathbb{B}^1} f(X^{t, \mathbf{x}}).\mathbf{v}] \quad (5.34)$$

$$\partial_{\mathbb{B}^1}^2 \varphi(t, \mathbf{x}).(\mathbf{v}, \mathbf{w}) = \mathbb{E}[\partial_{\mathbb{B}^1}^2 f(X^{t, \mathbf{x}}).(\mathbf{v}, \mathbf{w})], \quad (5.35)$$

for all $t \in [0, T]$, $\mathbf{x}, \mathbf{v}, \mathbf{w} \in \mathbb{B}^1(H)$. Finally, by (5.34), (5.35), boundedness of $\partial_{\mathbb{B}^1} f(X^{t, \cdot})(\omega)$ and $\partial_{\mathbb{B}^1}^2 f(X^{t, \cdot}(\omega))$, strong continuity of $\partial_{\mathbb{B}^1} f(X^{t, \cdot})(\omega)$ and $\partial_{\mathbb{B}^1}^2 f(X^{t, \cdot}(\omega))$, we obtain that $\varphi(t, \cdot)$ belongs to $\mathcal{G}^2(\mathbb{B}_\infty^1(H), \mathbb{R})$ and has bounded first and second order differentials. To conclude

the verification of Assumption 3.3(ii), it is sufficient to show that, for all $t \in [0, T]$, the maps

$$\begin{aligned} \mathbb{W} \times \mathbb{B}_{\sigma^s}^1(H) &\rightarrow \mathbb{R}, (\mathbf{x}, \mathbf{v}) \mapsto \partial_{\mathbb{B}^1} \varphi(t, \mathbf{x}).\mathbf{v} \\ \mathbb{W} \times \mathbb{B}_{\sigma^s}^1(H) \times \mathbb{B}_{\sigma^s}^1(H) &\rightarrow \mathbb{R}, (\mathbf{x}, \mathbf{v}, \mathbf{w}) \mapsto \partial_{\mathbb{B}^1}^2 \varphi(t, \mathbf{x}).(\mathbf{v}, \mathbf{w}) \end{aligned}$$

are sequentially continuous. This comes immediately by combining (5.32), (5.33), (5.34), (5.35), Proposition 5.6(ii), Proposition 5.7(i), Assumption 5.8(iii),(iv), the uniform boundedness of the differentials involved (we recall again Remark 5.4) and of the convergent sequences in $\mathbb{B}_{\sigma^s}^1(H)$.

Similarly, we can see that Assumption 3.3(iii) is verified by taking into account (5.32), (5.33), (5.34), (5.35), Proposition 5.6(iii), Proposition 5.7(ii), Assumption 5.8(iii),(iv), the uniform boundedness of the differentials involved and of the convergent sequences in $\mathbb{B}_{\sigma^s}^1(H)$. \blacksquare

Remark 5.10. Let $g: [0, T] \times H \rightarrow H$ be a continuous function, with $g(t, \cdot) \in C_b^2(H, H)$ and with differentials $D_H g, D_H^2 g$ uniformly continuous. Let us introduce the function \hat{b}^g defined by

$$\hat{b}^g(t, \mathbf{y}) := b \left(t, \int_{[0, T]} \tilde{g}(t-s, \tilde{\mathbf{y}}(t-s)) \mu(ds) \right) \quad \forall (t, \mathbf{y}) \in [0, T] \times \mathbb{B}^1(H),$$

where $\tilde{g}(r, x) := g(0, x)$ if $r < 0$. Consider the function

$$G: \mathbb{B}^1(H) \rightarrow \mathbb{B}^1(H), \mathbf{y} \mapsto \{g(t, \mathbf{y}(t))\}_{t \in [0, T]}.$$

Then G is well-defined, G belongs to $C_b^2(\mathbb{B}_\infty^1(H), \mathbb{B}_\infty^1(H))$, and $\hat{b}^g(t, \mathbf{y}) = \hat{b}(t, G(\mathbf{y}))$. By using these observations and the explicit expressions of DG, D^2G in terms of $D_H g, D_H^2 g$, it is not difficult to show that the results proved in this section can be extended to the case in which the drift \hat{b} in SDE (5.6) is replaced by the more general drift \hat{b}^g .

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